Lecture 1: Intro + 1D motion

[Syllabus]

Physics – perspective that the world around us is understandable.

→ From origins of universe to how our brains work

Laws of physics: mathematical relations between physical quantities

Dimensions		Units	(S.I.)
Length	→	meter	(m)
time	→	second	(s)
mass	→	Kilogram	(Kg)
temperature	→	Kelvin	(K)
⋮		⋮	

Conversion of units: "multiply by one, cleverly"

Ex: what is $25\,\frac{\text{miles}}{\text{hr}}$ in $\frac{m}{s}$?

$$\frac{25\,\cancel{\text{miles}}}{1\,\cancel{\text{hour}}}\cdot\frac{1\,\cancel{\text{hour}}}{3600\text{ seconds}}\cdot\frac{1609.34\text{ meters}}{1\,\cancel{\text{mile}}}\approx 11.2\,\frac{m}{s}$$

Scientific notation: write numbers as $a\cdot 10^n$, $1\le a<10$;

ex: 1 century $\approx (100\text{ years})\cdot(365\,\frac{\text{days}}{\text{year}})\cdot(24\cdot 3600\,\frac{\text{seconds}}{\text{day}})$

≈ 3153600000 seconds

$\approx 3.15\times 10^9$ seconds

(So, 1 nano-century is ~ 3 seconds)

Intro to Kinematics

Over the next several chapters:

Newtonian Mechanics

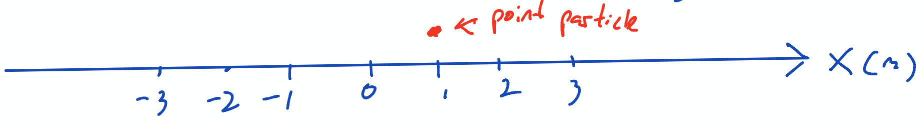
Kinematics
- positions
- velocities
- accelerations

"Describe motions w/ out considering causes"

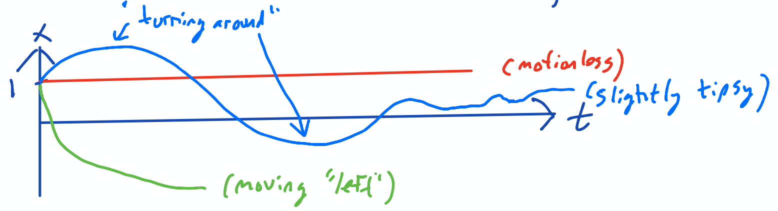
Dynamics
- Forces
- Torques
- Laws of Motion

1-D motion

Choose a coordinate system (arbitrary reference point, $x=0$, and units)

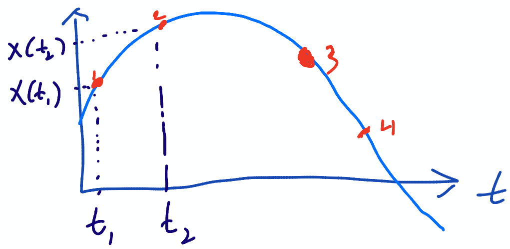


(1) Describe motion of abstract particle by writing its position as a function of time, $x(t)$



"x" is arbitrary – this could be left/right, or up/down, or...

(2) How to talk about how fast the particle is moving?



"Average velocity" between t_1 and t_2

$$V_{x,avg} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{\Delta x}{\Delta t} = \frac{\text{displacement}}{\text{time interval}}$$

displacement ≠ distance travelled!

distance > 0, but

Δx can be

- positive $x_2 - x_1 > 0$
- zero $x_3 - x_1 = 0$
- negative $x_4 - x_1 < 0$

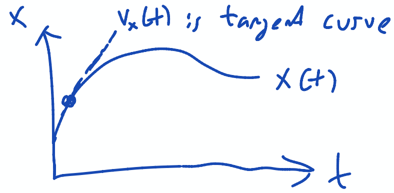
"average speed", $V_{avg} = \frac{\text{distance}}{\Delta t}$ ≠ average vel.

What if we measure two positions at closer time intervals?

"Instantaneous velocity"

$$V_x(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}$$

, the derivative

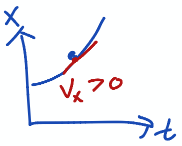


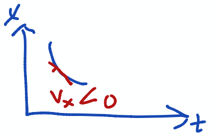
From calculus, there are simple cases:

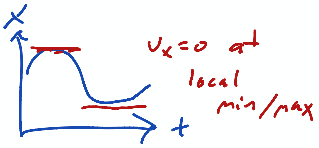
- $x(t) = \text{constant} \implies V_x = \frac{dx}{dt} = 0$
- $x(t) = a t^n \implies V_x = \frac{d}{dt}(a t^n) = a n t^{n-1}$

etc.

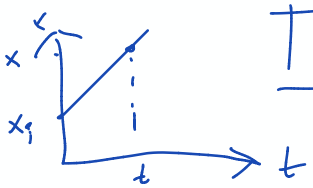
v_x can be :







Simple case : $v_x = \text{const}$



$$\boxed{x = x_i + v_x t}$$

example ! I'm walking in a straight line at $v_x = 10 \frac{m}{s}$

i.e., every second I move 10 meters; after 10 seconds I'm 100 m away (yep, apparently my walking speed is olympic level)

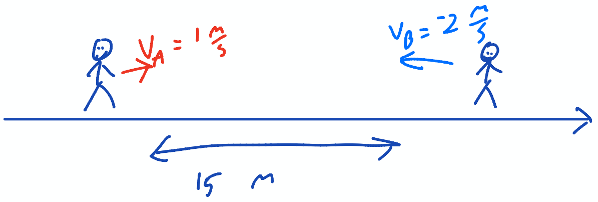
math : $x = x_i + v_x t$

$= x_i + 10 \frac{m}{s} \cdot 10 s$

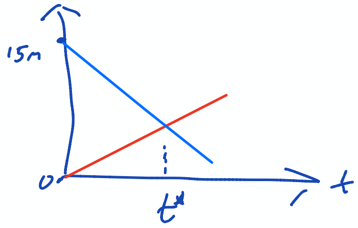
$= x_i + 100 m$

Example :

$t = 0$



When do A and B meet ?



$x_A = x_{A,i} + v_A t = 0 + t \cdot (1 \frac{m}{s})$

$x_B = x_{B,i} + v_B t = 15 m + t \cdot (-2 \frac{m}{s})$

$x_A(t^*) = x_B(t^*) \Rightarrow t^* = 15 - 2t^*$

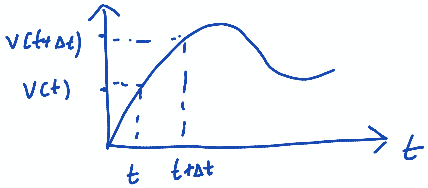
$\Rightarrow t^* = 5$ seconds

where? $x_A(t^*) = 5$ m

Lecture 2 : 1D motion, continued...

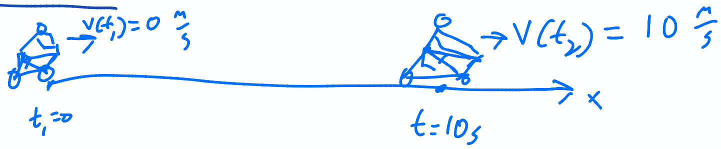
Last time : $x(t)$ and $V_x(t) = \frac{d}{dt}(x(t))$

Acceleration : change in $V_x(t)$ as a Function of time

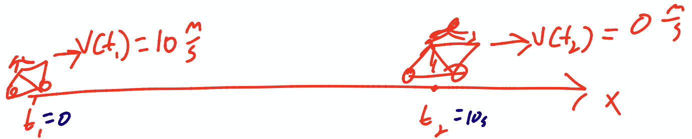


average acceleration $a_{avg} = \frac{\Delta V}{\Delta t}$, units of $\frac{m}{s^2}$

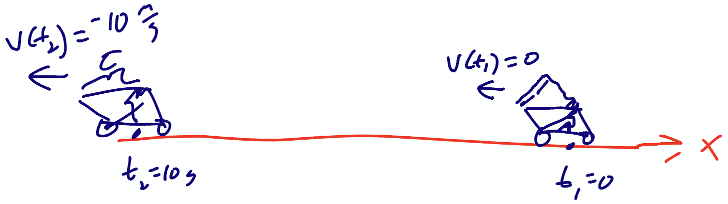
For instance :



$$a_{avg} = \frac{10-0}{10} = 1 \frac{m}{s^2}$$



$$a_{avg} = \frac{0-10}{10} = -1 \frac{m}{s^2}$$



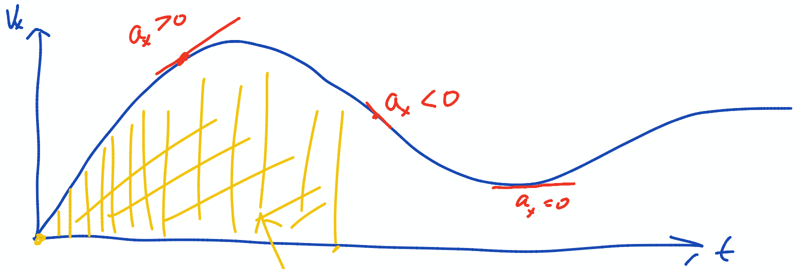
$$a_{avg} = -1 \frac{m}{s^2}$$

i.e : Accel can increase or decrease speed, etc.

Instantaneous acceleration

$$a_x(t) = \lim_{\Delta t \to 0} \frac{\Delta V_x}{\Delta t} = \frac{dV_x}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

5.



area is change in position in that time interval

$$\Delta x = \int_{t_1}^{t_2} V_x(t)\, dt$$

Kinematic structure:

$$\begin{tikzcd} x \arrow[r, bend left, "d/dt"] & V \arrow[l, bend left, "\int dt"] \arrow[r, bend left, "d/dt"] & a \arrow[l, bend left, "\int dt"] \end{tikzcd}$$

handy:

$$\frac{d}{dt}(t^n) = n t^{n-1}$$

Special case (but important/common!) : constant acceleration

(1) How does V change?

$$\frac{dV}{dt} = a$$

$$\Rightarrow \int_{t_1}^{t_2} \frac{dv}{dt}\, dt = a \int_{t_1}^{t_2} dt$$

$$\Rightarrow V(t_2) - V(t_1) = a(t_2 - t_1)$$

re-write: $V = V_i + a t$

(2) How does x change?

$$\frac{dx}{dt} = V = V_i + a t$$

$$\Rightarrow x = x_i + v_i t + \frac{a}{2} t^2$$

("3") Rearranging Kinematic expression.

Example: what if "time" isn't specified?

$$t = \frac{v - v_i}{a}$$

and $x = x_i + v_i t + \frac{1}{2} a t^2$ ... substitute:

$$\Rightarrow v^2 = v_i^2 + 2a(x - x_i)$$

Relevant application of "a = constant" scenario: Free-fall near earth's surface.

g ("accel. due to grav.") $\approx 9.8\,\frac{m}{s^2}$

Two options for coordinate system

$\uparrow y$

$\downarrow a = -g = -9.8\,\frac{m}{s^2}$

$$v = \begin{cases} \uparrow & (+) \\ \downarrow & (-) \end{cases}$$

$y = y_i + v_i t - \frac{gt^2}{2}$

$v = v_i - gt$

or

$y \downarrow$

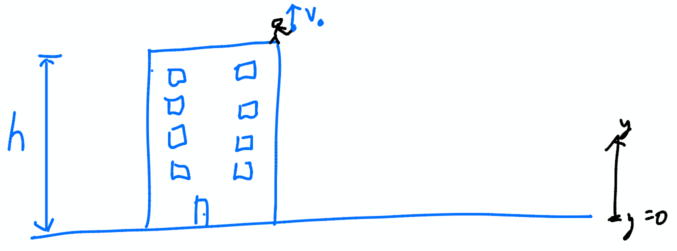
$\downarrow a = g = 9.8\,\frac{m}{s^2}$

$$v = \begin{cases} \uparrow & (-) \\ \downarrow & (+) \end{cases}$$

$y = y_i + v_i t + \frac{gt^2}{2}$

$v = v_i + gt$

Example: A student throws lab equipment upward over the side of building, w/ upward velocity v_0



(i) Time to reach max height?

$$v = 0 = v_0 - gt \quad \Rightarrow \quad t_{\text{max height}} = \frac{v_0}{g}$$

(ii) Height at maximum? (Several ways to solve... eg:

$$v^2 = 0^2 = v_0^2 - 2g(y - y_i)$$

$$\Rightarrow y_{max} = h + \frac{v_0^2}{2g}$$

(iii) When does it hit the ground?

$$y = 0 = h + v_0 t - \frac{gt^2}{2}$$

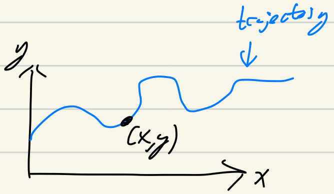
$$\Rightarrow t^2 - \frac{2v_0}{g}t - \frac{2h}{g} = 0$$

$$\Rightarrow t = \frac{v_0}{g} \boxed{\pm} \sqrt{\left(\frac{v_0}{g}\right)^2 + \frac{2h}{g}}$$

which sign do we pick? what does this mean?

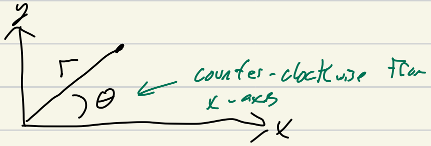
Lecture 3: Vectors and multidimensional motion

Vectors: Let's start in 2D



In 2D, can use (x,y), or

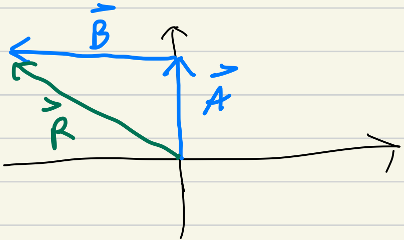
polar coordinates



$$r = \sqrt{x^2+y^2}$$

$$\tan\theta = y/x \;\; ; \;\; \overset{(0^\circ)}{0} \le \theta \le \overset{(360^\circ)}{2\pi}$$

Silly example: pirate treasure: walk North 200 paces then, on a heading of 179°, walk 312 paces



From this picture, we see we could have found the treasure by following $\vec R$:

$$\vec R = \vec A + \vec B$$

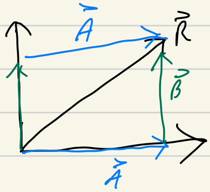
some physical quantities are scalar, some vector, eg.: ;

All of these are vectors: quantities with both a direction + a magnitude

$\vec R$ sometimes called "Resultant", draw tip-to-tail

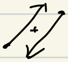
Vector arithmetic

- Sums are commutative : $\vec{R} = \vec{A} + \vec{B} = \vec{B} + A$



and associative : $\vec{R} = (\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$

- $\vec{A} + (-\vec{A}) = 0$

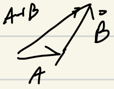

$= 0$

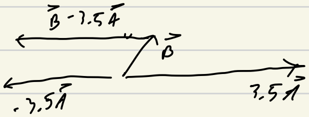
↖ vector of same magnitude as $\vec{A}$, but opposite orientation

- $m\vec{A} = \begin{cases} \text{vector of magnitude } |m| \cdot A \text{ in direction of } \vec{A} & \text{if } m > 0 \\ \text{" " " " " " in opposite direction of } \vec{A} & \text{if } m < 0 \end{cases}$

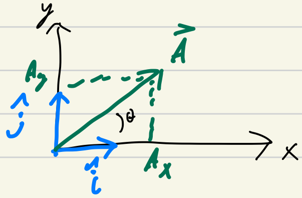
- Subtracting vectors → add negative vectors

e.g. $\vec{B} - 3.5\vec{A} = \vec{B} + (-3.5\vec{A})$





Vector components



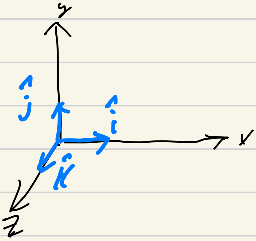
$\vec{A} = A_x \hat{i} + A_y \hat{j}$,

$\hat{i}$ and $\hat{j}$ are unit vectors

$|\hat{i}| = |\hat{j}| = 1$

$A = |\vec{A}| = \sqrt{A_x^2 + A_y^2}$; $\tan\theta = \frac{A_y}{A_x}$

in 3D :



$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

Vector addition is component-wise:

if $\vec{A} = A_x \hat{i} + A_y \hat{j}$ and $\vec{B} = B_x \hat{i} + B_y \hat{j}$

$$\vec{R} = \vec{A} + \vec{B} = A_x \hat{i} + A_y \hat{j} + B_x \hat{i} + B_y \hat{j}$$

assoc. + commut.

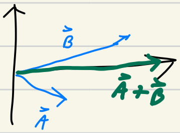
$$= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$

$$= R_x \hat{i} + R_y \hat{j}$$

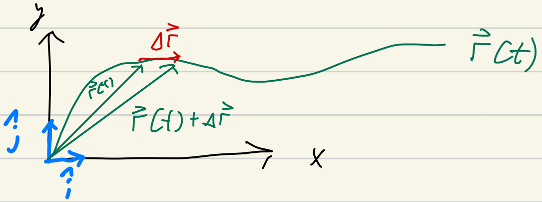
$$|\vec{R}| = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

ex: $\vec{A} = 2\hat{i} - \hat{j}$, $B = 5\hat{i} + \hat{j}$

$R = 7\hat{i}$



Motion in 2D



$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

$$\vec{r}(t)+\Delta\vec{r} = (x(t)+\Delta x)\,\hat{i} + (y(t)+\Delta y)\,\hat{j}$$

$$\vec{V}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \left(\frac{\Delta x}{\Delta t}\right)\hat{i} + \left(\frac{\Delta y}{\Delta t}\right)\hat{j}$$

take limit $\Delta t \to 0$: $\vec{V} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\hat{i} + \frac{dy}{dt}\,\hat{j}$

(graphically : tangent vector to trajectory)

Similarly :

$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt}$ ... each component indep.!

For instance, take constant $\vec{a} = a_x\,\hat{i} + a_y\,\hat{j}$

$= \frac{dv_x}{dt}\,\hat{i} + \frac{dv_y}{dt}\,\hat{j}$

ie, $\begin{cases} \frac{dv_x}{dt} = a_x \\ \frac{dv_y}{dt} = a_y \end{cases} \Rightarrow \begin{array}{l} v_x = v_{x,i} + a_x t \\ v_y = v_{y,i} + a_y t \end{array}$

likewise, position :

$$x = x_i + v_{x,i}\,t + \frac{a_x}{2}\,t^2$$

$$y = y_i + v_{y,i}\,t + \frac{a_y}{2}\,t^2$$

or, in vector notation

$$\vec{v} = \vec{v}_i + \vec{a}t$$

$$\vec{r} = \vec{r}_i + \vec{v}_i t + \frac{\vec{a}}{2}t^2$$

in any dimension !

Application : tomato projectile

i.e., $V_{i,x} = V_i \cos\theta$; $V_{y_i} = V_i \sin\theta$; assume $0 < \theta < \frac{\pi}{2}$

$x_i = 0$, $y_i = h$

$\vec{a} = -g\,\hat{j}$

Handle each coordinate separately;

horizontal motion : $a_x = 0 \Rightarrow V_x = \text{const};$

$$x(t) = (V_i \cos\theta)\, t$$

Vertical motion : $V_y(t) = V_i \sin\theta - gt$

$$y(t) = (V_i \sin\theta)\, t - \frac{g}{2} t^2$$

(1) time to reach highest point (just like last class!)

$$V_y = 0 = V_i \sin\theta - g t_A$$

$$\Rightarrow t_A = \frac{V_i \sin\theta}{g}$$

By symmetry, time to reach B is $t_B = 2t_A$

(2) velocity at B ?

$$\vec{V}_B = \vec{V}_i - g t_B\,\hat{j} = V_i \cos\theta\,\hat{i} + V_i \sin\theta\,\hat{j} - 2\frac{V_i \sin\theta}{g} \cdot g\,\hat{j}$$

$$= V_i \cos\theta\,\hat{i} - V_i \sin\theta\,\hat{j}$$

(3) given h, R, V_i, how to choose θ ? Left as exercise!

Lecture 4: 2D motion, continued.

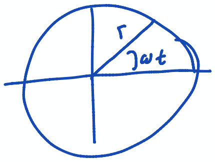
Last time: vector Kinematics: $\vec{r} = x(t)\,\hat{i} + y(t)\,\hat{j}$

$$\vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a} = \frac{d\vec{v}}{dt}$$

Circular motion:

Suppose $\vec{r}(t) = r\left(\cos(\omega t)\,\hat{i} + \sin(\omega t)\,\hat{j}\right)$ ... what does this look like?

- Notice $|\vec{r}(t)|^2 = r^2\left[\cos^2\omega t + \sin^2\omega t\right] = r^2$ ... particle is always same distance from the origin



- How long to go in a circle?

$$\omega T = 2\pi \quad \text{(radians)} \Rightarrow$$

"angular speed" "period"

$$\omega = \frac{2\pi}{T} = 2\pi f$$

"Freq."

- Velocity?

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left[r\cos(\omega t)\,\hat{i} + r\sin(\omega t)\,\hat{j}\right]$$

$$= -r\omega\,\sin(\omega t)\,\hat{i} + r\omega\,\cos(\omega t)\,\hat{j}$$

- Speed? $|\vec{v}| = r\omega$: constant speed

$$= \frac{2\pi r}{T}$$ ← distance around circle

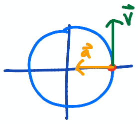
← period

- acceleration ?

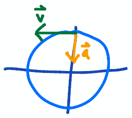
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$$= -r\omega^2\left[\cos(\omega t)\hat{i} + \sin(\omega t)\hat{j}\right]$$

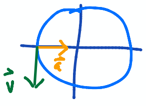
$$= -\omega^2\ \vec{r}$$



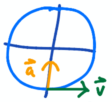
$t = 0$



$t = \frac{T}{4}$



$t = \frac{T}{2}$



$T = \frac{3T}{4}$

here $\vec{a}$ points to center of circular trajectory : Centripetal acceleration , $\vec{a}_c$

↑

"center-seeking"

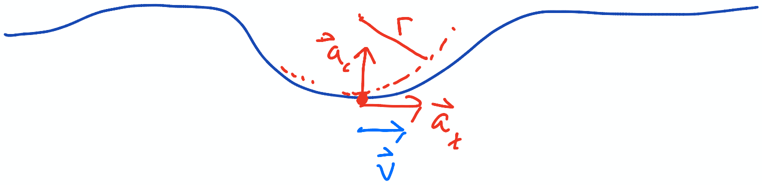
$$a_c = \omega^2 r = \frac{v^2}{r}$$

We have acceleration $\neq 0$ and constant speed

⇒ velocity vector (here, direction) is changing

(planetary motion; car going around a curve, charged particle in a field)

Suppose a particle moves on a curved path:



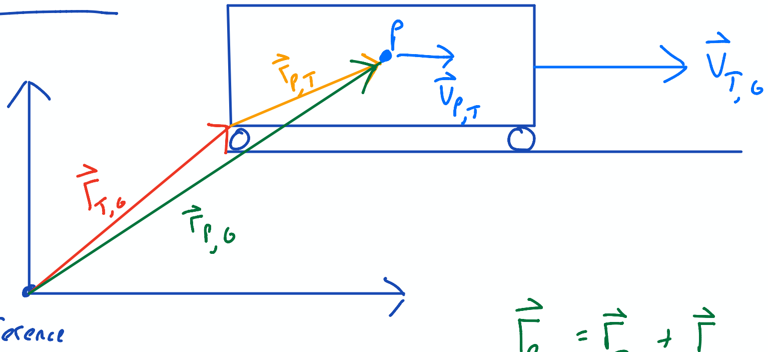
total accel $\vec{a} = \vec{a}_c + \vec{a}_t$

$\vec{a}_c$ ↑ changes $\vec{v}$ direction

$\vec{a}_t$ ↖ changes $\vec{v}$ magnitude

---

relative motion



$$\vec{r}_{P,G} = \vec{r}_{T,G} + \vec{r}_{P,T}$$

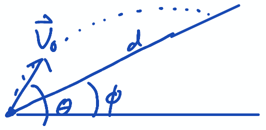
$$\frac{d\vec{r}_{P,G}}{dt} = \frac{d\vec{r}_{T,G}}{dt} + \frac{d\vec{r}_{P,T}}{dt} = \vec{V}_{TG} + \vec{v}_{PT} = \vec{V}_{P,G}$$

$$\frac{d\vec{v}_{PG}}{dt} = \frac{d\vec{V}_{TG}}{dt} + \frac{d\vec{v}_{PT}}{dT}$$

if $\vec{V}_{T,G}$ is constant

$$\vec{a}_{PG} = \vec{a}_{PT}$$

Worked example :



How far up ramp does projectile land?

projectile : $\vec{r} = \vec{r}_i + \vec{v}_i t + \frac{\vec{a}}{2} t^2$

$$= 0 + t\left[V_0 \cos\theta\,\hat{i} + V_0 \sin\theta\,\hat{j}\right] - \frac{g}{2} t^2 \hat{j}$$

ramp : $d\cos\phi\,\hat{i} + d\sin\phi\,\hat{j}$

when? (equate x-positions) $V_0 t \cos\theta = d\cos\phi \Rightarrow t = \frac{d\cos\phi}{V_0\cos\theta}$

where (equate y-positions)

$$V_0 t \sin\theta - \frac{g}{2} t^2 = d\sin\phi$$

$$\Rightarrow \frac{V_0 d\cos\phi\sin\theta}{V_0\cos\theta} - \frac{g}{2}\left(\frac{d^2\cos^2\phi}{V_0^2\cos^2\theta}\right) = d\sin\phi$$

$$\Rightarrow V_0\cos\phi\sin\theta - \frac{g}{2}\frac{d\cos^2\phi}{V_0\cos\theta} = V_0\cos\theta\sin\phi$$

$$\Rightarrow d = \frac{V_0\boxed{(\cos\phi\sin\theta - \cos\theta\sin\phi)}}{g\cos^2\phi} \quad 2V_0\cos\theta$$

For $(\cos\phi\sin\theta - \cos\theta\sin\phi)$ use trig identities

$\left(\cos x = \frac{e^{ix} + e^{-ix}}{2}\right.$

$\left.\sin x = \frac{e^{ix} - e^{-ix}}{2i}\right)$

to simplify

$$\Rightarrow d = \frac{2V_0^2\cos\theta}{g\cos^2\phi}\cdot\sin(\theta - \phi)$$

Lecture 5 : Newton's Laws (Part 1)

[Announce HW 4 – encourage early start / Phys Mentor Help]

1st Law (The Law of inertia) : An object which does not experience a net (total) Force moves at constant velocity

- $\vec{v} = 0$ counts!
- "object at rest tends to stay at rest; an object in motion tends to stay in motion"
  - → Different From our everyday experience [Friction] ... revolutionary concept.
- This law applies to "inertial" (non-accelerating) Frames. To a good approximation, some non-inertial Frames obey it (e.g.: Earth)

2nd Law : The net Force (or vector sum of Forces, $\sum \vec{F}$) on an object causes an acceleration proportional to that net Force :

$$\boxed{\vec{a} = \frac{\vec{F}}{m}}$$

or, $\vec{F} = m\vec{a}$

- Constant of proportionality is called mass : Fundamentally mass represents resistance of a body to changes in its acceleration.

Types of common Forces : "contact", elastic, tensions,

" Fundamental Forces : $\underbrace{\text{gravity, electromagnetic, weak, strong,}}_{}$ all everyday Forces come From those.

3rd Law : (Actio- Reactio) :

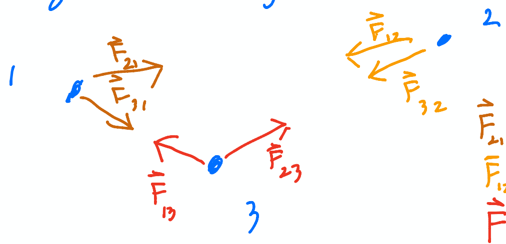
$$\underbrace{\vec{F}_{ab}}_{\text{Force exerted on b by a}} = -\underbrace{\vec{F}_{ba}}_{\text{Force exerted on a by b}}$$

E.g. : pushing chalk on board; standing on ground.
(step off skateboard; jump on Earth]

"Free body diagrams" : If we know all forces on an object, we know $\vec{a}$ (and, hence, all kinematics).

Often helps to draw free-body diagrams, isolating all forces on each object.

E.g. : 3 gravitational objects

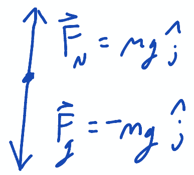


$$\vec{F}_{21} + \vec{F}_{31} = m\vec{a}_1$$
$$\vec{F}_{12} + \vec{F}_{32} = m\vec{a}_2$$
$$\vec{F}_{13} + \vec{F}_{23} = m\vec{a}_3$$

Example 1 :

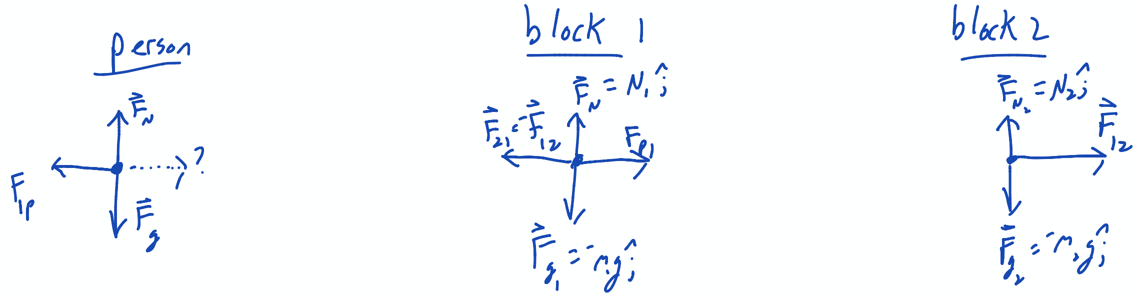
mass at rest

FBD



$$\vec{F}_N = mg\,\hat{j}$$
$$\vec{F}_g = -mg\,\hat{j}$$

example 2 (pushing 2 blocks horizontally)



Analysis: all y-Forces are balanced, so $a_y = 0$ For all objects

Block 1: $\Sigma F_x = \vec{F}_{p_1} - \vec{F}_{12} = m_1 a_x$

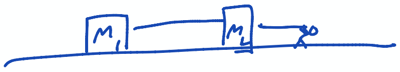
+ **Block 2:** $\Sigma F_x = \vec{F}_{12} = m_2 a_x$

blocks moving together ⇒ same accel.

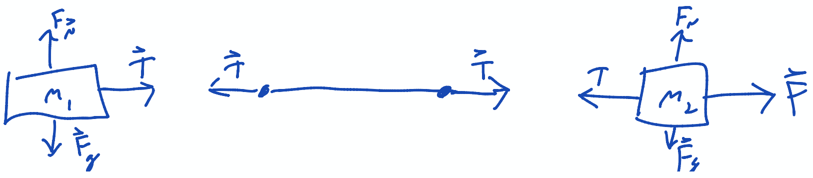
$$\Rightarrow \quad \vec{F}_{p_1} = (m_1 + m_2) a_x$$

$$\Rightarrow \quad a_x = \frac{F_{p_1}}{m_1 + m_2} \; ; \quad \vec{F}_{12} = m_2 \frac{F_{p_1}}{m_1 + m_2}$$

Example 3: 2 block with ideal (massless, unbreakable) rope:



Analysis: again, net Forces only in x-direction.



Block 1 : $T = m_1 a_x$

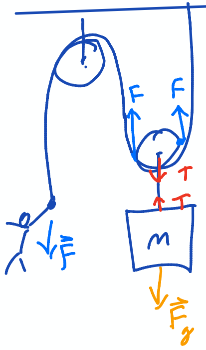
+ Block 2 : $F - T = m_2 a_x$

$$F = (m_1+m_2)a_x \Rightarrow a_x = \frac{F}{m_1+m_2}$$

Tension: $T = m_1 a_x = m_1 \frac{F}{m_1+m_2}$

---

example 4 a pulley system just at rest



Forces on block:

$T - mg = a_x = 0 \Rightarrow T = mg$

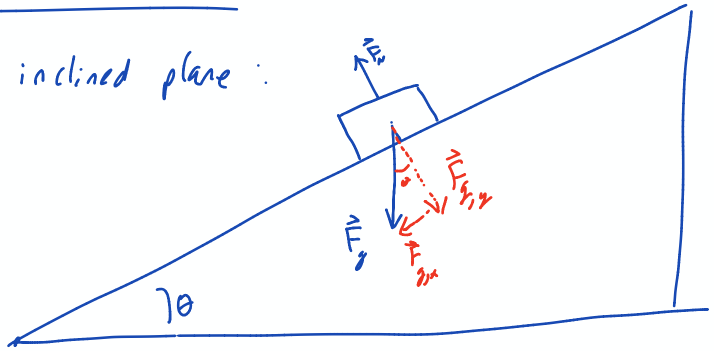
Forces on pulley

$2F - T = a_x = 0$

$\Rightarrow F = \frac{mg}{2}$

---

example 5 inclined plane :



$\vec F_g = \vec F_{g,x} + \vec F_{g,y}$; $F_{g,x} = mg\sin\theta$

$F_{g,y} = mg\cos\theta$

perpendicular to plane : $\Sigma \vec F_y = \vec F_N - mg\cos\theta = a_y = 0 = |F_N| = mg\cos\theta$

along plane : $\Sigma F_x = \vec F_{g,x} = mg\sin\theta$

$\Rightarrow a_x = g\sin\theta$

limits?

$\theta = 0 \Rightarrow a_x = 0$ ✓

$\theta = \frac{\pi}{2} \Rightarrow a_x = g$ ✓

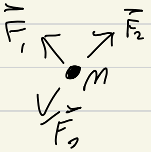
9/13/22

Newton's Laws, part 2

[Announcements]

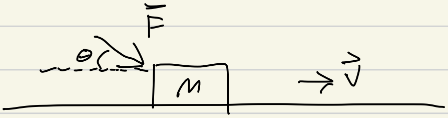
Last time: No Force ⇒ const velocity (Law of Inertia)

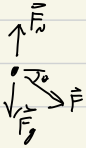
$\vec{F}_{ab} = -\vec{F}_{ba}$ (reciprocity of Forces)



$$\therefore \sum_i \vec{F}_i = m\vec{a}$$

Example: Angled Force on block, Frictionless surface:





$\sum \vec{F} = m\vec{a}$: $\sum F_y = 0 \Rightarrow |F_N| = mg + |F| \sin\theta$

$\sum F_x = ma_x \Rightarrow |F| \cos\theta = ma_x$

So : $a_x = \frac{F\cos\theta}{m}$

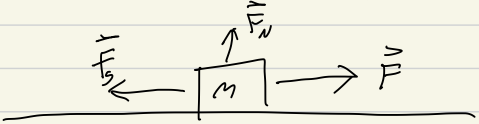
$$v_x(t) = v_{i,x} + \frac{F\cos\theta}{m} t$$

$$x(t) = x_i + v_{i,x} t + \frac{1}{2} \frac{F\cos\theta}{m} t^2$$

Friction : opposing relative motion of surfaces in contact

We kind of know how friction works, but researchers still working on details!

Static Friction : Suppose you push on an object at rest, but it doesn't move... by Newton's 2nd Law, there *must* be an opposing Force, so that $\vec{F}_{net} = 0$



(sliding friction)

Rules for static Friction: (1) acts parallel to surface

(2) magnitude depends on the two surfaces

(3) Can only support a maximum amount of opposing Force:

$$F_s \le \mu_s |\vec{F}_N|$$

Normal Force

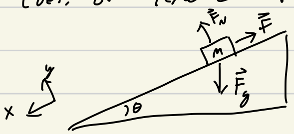
coef. of static Friction

Kinetic Friction: Again, a Force that opposes relative motion of surfaces, so direction of $\vec{F}_K$ is opposite to motion

$$|F_K| = \mu_K |\vec{F}_N|$$

coef. of Kinetic Friction, $\mu_K < \mu_s$

Example: Inclined plane w/ Friction



(1) Angle at which block is on verge of slipping?

Balance 'x' Forces: $mg\sin\theta = F_{s,max} = \mu_s|\vec{F}_N| = \mu_s mg\cos\theta$

$\Rightarrow \tan\theta = \mu_s$

So: if $0 \le \theta \le \tan^{-1}\mu_s$, static Friction holds the block.

(2) If the block is sliding down, what is its accel?

$\sum F_x = ma_x \Rightarrow mg\sin\theta - \mu_K mg\cos\theta = ma_x$

$\Rightarrow a_x = g(\sin\theta - \mu_K \cos\theta)$

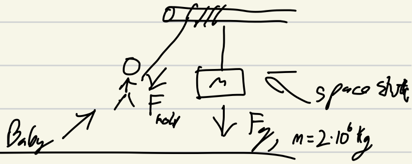
Hard example: ropes and Friction

- Friction messes up our "ideal ropes have T the same everywhere" approximation!, even if massless
- [This is how a light person can belay a heavy person while rock climbing, e.g.]

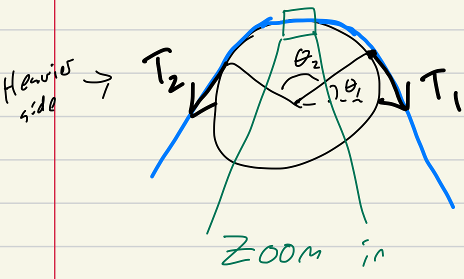
Need to carefully consider how both T and F_N changes as a rope is in contact w/ a surface.

estimation of # of turns for baby to hold space shuttle?

5 turns ?
500 turns ?
50000 turns ?
5000000 turns ?

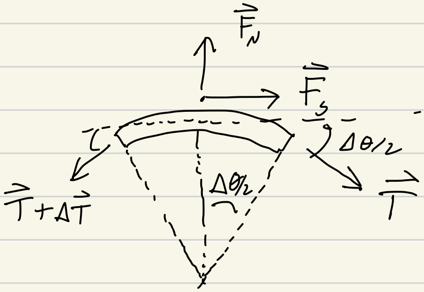


Front view



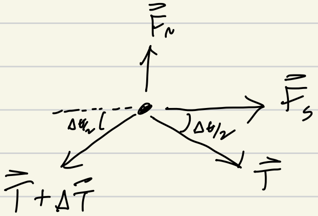
, with some μ_s

infinitesimally to a section of rope:



Hard example, cont.

FBD for section of rope:



rope not moving $\Rightarrow$ $\vec{F}_{net} = 0$

How big can this be? $\mu_s|\vec{F}_N|$

In x-direction

$$\sum F_x = 0 = T\cos\frac{\Delta\theta}{2} - (T+\Delta T)\cos\frac{\Delta\theta}{2} + \boxed{F_s}$$

$$= -\Delta T\cos\frac{\Delta\theta}{2} + \mu_s|\vec{F}_N|$$

In y-direction: $\sum F_y = 0 = |\vec{F}_N| - T\sin\frac{\Delta\theta}{2} - (T+\Delta T)\sin\frac{\Delta\theta}{2}$

$$\Rightarrow |\vec{F}_N| = 2T\sin\frac{\Delta\theta}{2} - \Delta T\sin\frac{\Delta\theta}{2}$$

Back to x-direction:

$$0 = -\Delta T\cos\frac{\Delta\theta}{2} + \mu_s 2T\sin\frac{\Delta\theta}{2} - \mu_s\Delta T\sin\frac{\Delta\theta}{2}$$

Let's now assume both $\Delta\theta$ and ΔT are small!

$\sin\frac{\Delta\theta}{2} \approx \Delta\theta/2$

$\cos\frac{\Delta\theta}{2} \approx 1$

] Taylor series

So: $0 \approx -\Delta T + \mu_s\cdot 2T\frac{\Delta\theta}{2}\boxed{-\mu_s\Delta T\frac{\Delta\theta}{2}} \rightarrow$ very small

$$\Rightarrow \frac{\Delta T}{T} = \mu_s\Delta\theta$$

cont,

That's true in every little section of rope! Integrating:

$$\int_{T_1}^{T_2} \frac{dT}{T} = \int_{\theta_{start}}^{\theta_{stop}} \mu_s \, d\theta$$

$$\Rightarrow \ln T_2 - \ln T_1 = \mu_s (\theta_{stop} - \theta_{start})$$

$$\Rightarrow \ln \frac{T_2}{T_1} = \mu_s \text{``}\Delta\theta\text{''}$$

$$\Rightarrow T_2 / T_1 = \exp(\mu_s \Delta\theta)$$

in the context of our example:

$$T_{load} = T_{hold} \, e^{\mu_s \Delta\theta}$$

if $\mu_s \approx 0.4$ and $T_{load} = (2 \cdot 10^6 \, kg) \cdot g$

and $T_{hold} = (1 kg) \cdot g$

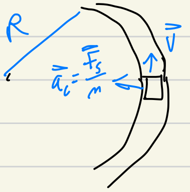
we need $\Delta\theta \approx 36$ radians

≈ 5.7 complete turns...

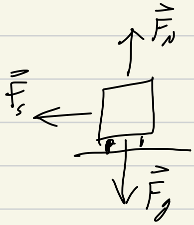
[hope that rope + bus is strong!]

Easier example: on a Flat road, what is max speed around a turn?

top view



side view



centripetal accel for constant speed? $a_c = \omega^2 R = \frac{v^2}{R}$

Friction: $\mu_s |\vec{F}_N| \leq mg\mu_s$

$$\Rightarrow \cancel{m} g \mu_s = \cancel{m} \frac{v_{max}^2}{R}$$

$$\Rightarrow v_{max} = \sqrt{R g \mu_s}$$

[Real life: banked roads have a component of Normal Force that provides some of the needed centripetal accel...
How much? Which is more important?]

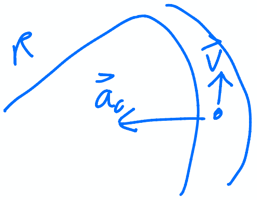
Lecture 7

- Next HW posted on webassign;
- Practice problems posted on webassign (!)
- Survey results: good balance on speed/difficulty/material (Themes)
- Responses to Q's
  - Exam structure
  - More examples / guided problem solving
  - Formulas / algebra happen too fast
  - More practice problems
  - Better handwriting (I'm so sorry!)
  - Positive class energy (thanks!)

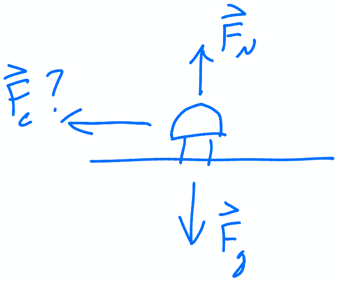
Today: centripetal Forces, Friction + motion

Q: On a flat road, what is the max speed you can turn in a circle of radius R?

Top view



Side view



Circular motion : $\vec{r}(t) = r\left[\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}\right]$

$$\vec{a}(t) = -r\omega^2\,\vec{r}(t)$$

so $|\vec{a}_c| = \omega^2 R$

Newton's 2^{nd} Law : $\vec{a} = \frac{\vec{F}}{m}$

What provides the Force? Friction!

Max static Friction?

$$\vec{F}_s \le \mu_s |\vec{F}_N| = \mu_s mg$$

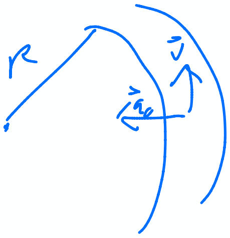
So : $m|\vec{a}_c| = m\,\omega_{max}^2 R = \mu_s mg$

$$\Rightarrow m\frac{v_{max}^2}{R} = \mu_s mg \Rightarrow v_{max}^2 = \mu_s Rg$$

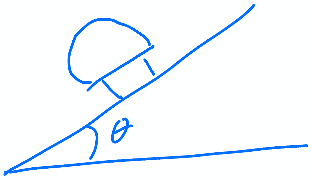
$$v_{max} = \sqrt{\mu_s Rg}$$

Real life: banked roads!

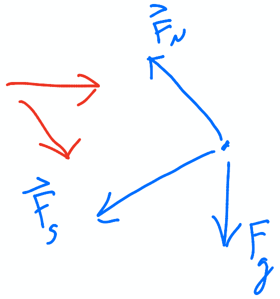
top view



side view



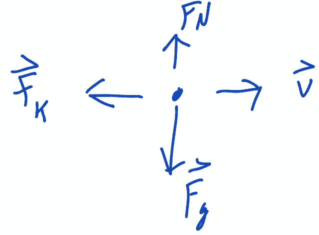
two forces can point towards center of turn!



Motion w/ resistive Forces (Friction, drag, etc.)

Suppose I kick something so that it starts moving with velocity $\vec{v}_0$ ... how does it slow down?

Case 1 : Friction



Newton's law : $|F_N| = mg$; $\vec{F}_K = \mu_K mg$

So: $ma_x = -\mu_K mg \Rightarrow \underbrace{a_x = -\mu_K g}_{}$, a constant

$$\vec{v}(t) = \vec{v}_0 + \vec{a}\,t$$

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \tfrac{1}{2}\vec{a}\,t^2$$

distance traveled?

$$t_{stop} = \frac{v_0}{a_x} = \frac{v_0}{\mu_K g}$$

$$(r - r_0) = v_0 \cdot \frac{v_0}{\mu_K g} + \frac{1}{2}(-\mu_K g)\left(\frac{v_0^2}{\mu_K^2 g^2}\right)$$

$$= \frac{v_0^2}{\mu_K g} - \frac{1}{2}\frac{v_0^2}{\mu_K g} = \frac{1}{2}\frac{v_0^2}{\mu_K g}$$

(is there another kinematic eqn? $v^2 = v_0^2 + 2a(x - x_0)$

Case 2: Drag proportional to speed ("viscous drag")

Newton's law: $\vec{a} = \frac{\vec{F}}{m}$

$$\frac{d\vec{v}}{dt} = \frac{-\gamma}{m}\vec{v} \qquad \text{differential equation!}$$

$$\Rightarrow \vec{v}(t) = \vec{v}_0\, e^{-\frac{\gamma}{m}\cdot t}$$

Check: $\frac{d\vec{v}}{dt} = \vec{v}_0 \cdot \frac{-\gamma}{m} e^{-\frac{\gamma}{m}t} = \frac{-\gamma}{m}\vec{v}$ ✓

$$(\vec{r} - \vec{r}_0) = \int_0^t v(t')\,dt'$$

$$= \left[\frac{-v_0 m}{\gamma} e^{-t'\gamma/m}\right]_0^t$$

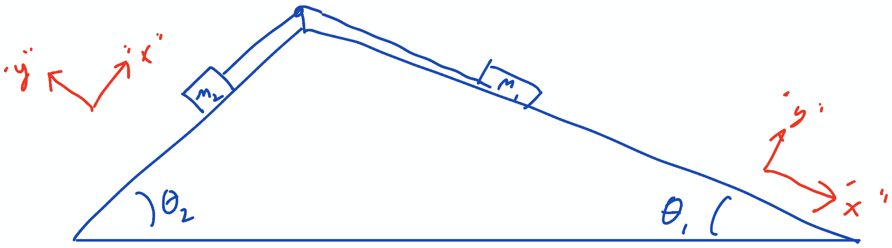
$$= \underbrace{\frac{v_0 m}{\gamma}}_{\text{magnitude of distance eventually traveled.}} \underbrace{\left[1 - e^{-\gamma t/m}\right]}_{\text{approach to final position}}$$

Case 3: drag proportional to $(\text{speed})^2$ ("inertial drag")

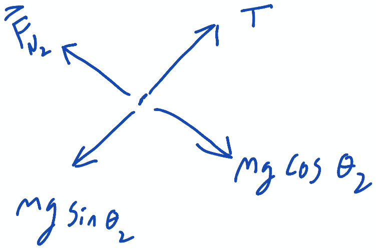
[in textbook!]

Other Examples (setting up, physically)

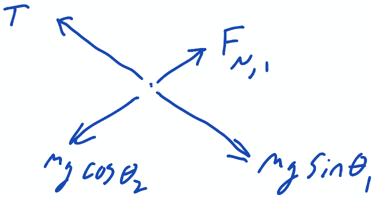
2 blocks, 2 angles, no Friction



mass 2



mass 1

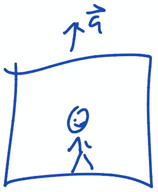


In direction of each block's plane:

$$mg\sin\theta_1 - T = m_1 a_1$$

$$T - mg\sin\theta_2 = m_2 a_2 \quad , \quad a_1 = a_2$$

The elevator:



$$m\vec{a} = \vec{F}_N + \vec{F}_g$$

$$\Rightarrow ma_y = F_N - mg$$

$$\Rightarrow F_N = m(a_y + g)$$

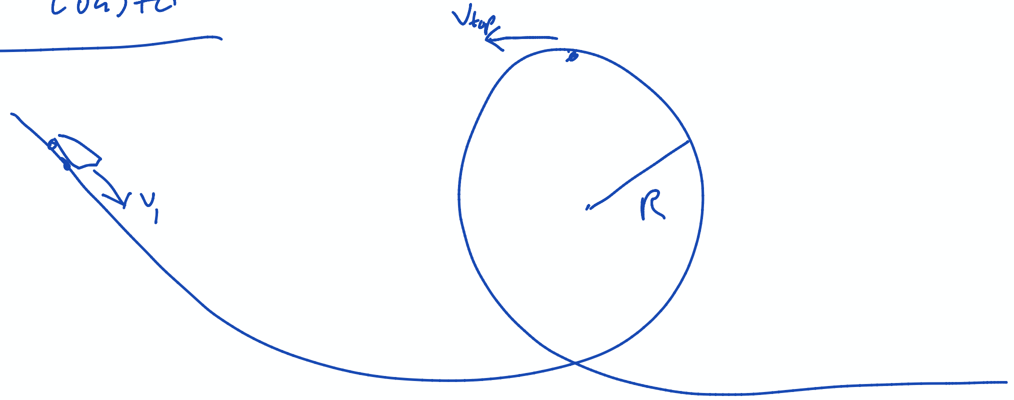
How hard you feel the Floor pushing on you!

$a > 0$: $F_N > mg$ ("appears heavy")

$a < 0$ $F_N < mg$ ("appears light")

$a = -g$ $F_N = 0$ ("appears weightless")

Roller coaster



minimum velocity at top to keep moving in circle?

$\vec{F}_N \downarrow \downarrow \vec{F}_g$

There must be a centrip. Force / accel :

$$\vec{F}_c = \vec{F}_N + F_g$$

$$\frac{mv^2}{R} = F_N + F_g \Rightarrow F_N = \frac{mv^2}{R} - mg$$

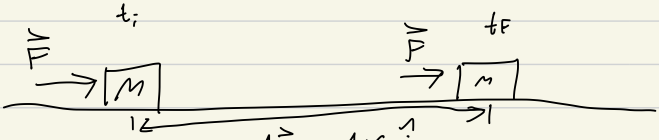
$$|F_N| \geq 0 = \frac{v^2}{R} - g \geq 0$$

$$\Rightarrow v \geq \underbrace{\sqrt{Rg}}_{\downarrow \text{ min. speed!}}$$

Lecture 8: Energy of a system

[Logistics?]

Suppose we push a block on a frictionless surface, constant F, over some distance



$$\Delta \vec{r} = \Delta x \,\hat{i}$$

$$\vec{F} = F \,\hat{i}$$

F constant $\Rightarrow a = \frac{F}{m}$ is constant,

$$\Rightarrow v_f^2 = v_i^2 + 2a(\Delta x) = v_i^2 + 2\frac{F}{m}\Delta x$$

rearrange: $\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = F \cdot \Delta x$

Let's use this example to motivate some definitions:

Kinetic Energy: $K = \frac{mv^2}{2}$

(change in KE: $\Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$)

Work done by $\vec{F}$ on system: $W_{external} = \underbrace{F_{external} \cdot \Delta X}_{\text{Units: Newtons}\cdot\text{meters} \;=\; \text{Joules}}$

In 2D: For a constant force, $\vec{F}$, acting over a displacement $\Delta\vec{r}$:

$$\vec{F} = F_x\,\hat{i} + F_y\,\hat{j} \quad ; \quad \Delta\vec{r} = \Delta x\,\hat{i} + \Delta y\,\hat{j}$$

$$K = m\vec{v}^{\,2}/2 = \frac{m}{2}\left(v_x^2 + v_y^2\right)$$

$$\Delta K = \frac{m}{2}\vec{v}_f^{\,2} - \frac{m}{2}\vec{v}_i^{\,2} = \frac{m}{2}\left(v_{x,f}^2 + v_{y,f}^2\right) - \frac{m}{2}\left(v_{x,i}^2 + v_{y,i}^2\right)$$

$$= \underbrace{\left(\frac{m}{2}v_{x,f}^2 - \frac{m}{2}v_{x,i}^2\right)}_{F_x\,\Delta x} + \underbrace{\left(\frac{m}{2}v_{y,f}^2 - \frac{m}{2}v_{y,i}^2\right)}_{F_y\,\Delta y}$$

$$\Rightarrow \Delta K = \vec{F}\cdot\Delta\vec{r} = F_x\,\Delta x + F_y\,\Delta y$$

Scalar product of vectors

(multiply vector by vector to get scalar)

a.k.a. the "dot product"

Suppose, in Cartesian coordinates,

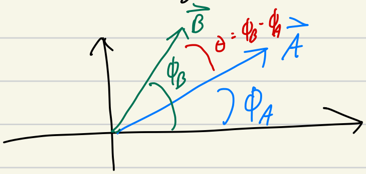
$$\vec{A} = (A_x, A_y, A_z) \qquad (\text{i.e., } \vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k})$$
$$\vec{B} = (B_x, B_y, B_z)$$

$$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = \vec{B}\cdot\vec{A}}$$

← definition of scalar product

Note: $\vec{A}\cdot\vec{A} = A_x^2 + A_y^2 + A_z^2 = |\vec{A}|^2$

Note: let



$$\vec{A} = (\overbrace{|\vec{A}|\cos\phi_A}^{A_x}, \overbrace{|\vec{A}|\sin\phi_A}^{A_y})$$

$$\vec{B} = (\underbrace{|\vec{B}|\cos\phi_B}_{B_x}, \underbrace{|\vec{B}|\sin\phi_B}_{B_y})$$

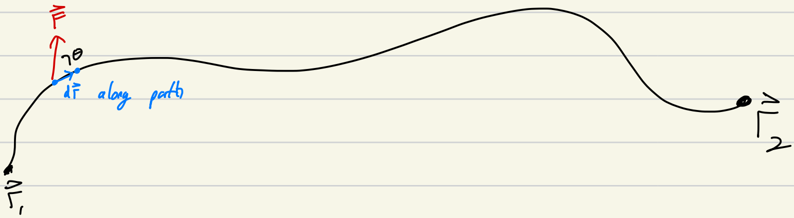
$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y = |\vec{A}|\cdot|\vec{B}|\left(\cos\phi_A\cos\phi_B + \sin\phi_A\sin\phi_B\right)$$

$$= |\vec{A}|\cdot|\vec{B}|\cdot\cos(\phi_B - \phi_A)$$

$$\Rightarrow \boxed{\vec{A}\cdot\vec{B} = |\vec{A}|\cdot|\vec{B}|\cdot\cos\theta}$$

← geometric interpretation of dot product

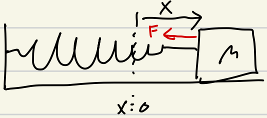
Calculus version of work done along a path:

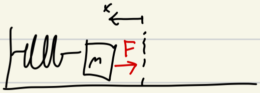


Add up contribution from force along whole path:
each infinitesimal work: $dW = \vec{F}\cdot d\vec{r}$

$$\Delta K = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_1^2 F_x\,dx + \int_1^2 F_y\,dy$$

Example: A spring has elastic force $F(x) = -\overbrace{k}^{\text{spring constant}}x$



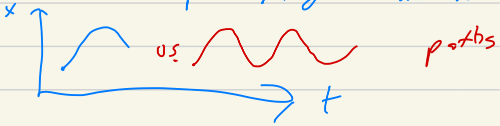


What work does the spring do as block moves from one position to another?

$$W_s = \int_{x_1}^{x_2} F(x)\,dx = \int_{x_1}^{x_2} (-kx)\,dx = \left[-\frac{kx^2}{2}\right]_{x_1}^{x_2} = \underbrace{\frac{kx_1}{2} - \frac{kx_2}{2}}$$

does not depend on path!

Think about positive/negative work for



How does KE of block change?

$U_s(x) = \frac{kx^2}{2}$: "elastic potential energy"

$F_s(x) = -\frac{dU_s}{dx} = -kx$

a "conservative" Force : work by Force is indep. of path

$\Delta K = W_s$

$$\Rightarrow \frac{mv_2^2}{2} - \frac{mv_1^2}{2} = \frac{kx_1^2}{2} - \frac{kx_2^2}{2}$$

$$\Rightarrow \frac{mv_2^2}{2} + \frac{kx_2^2}{2} = \frac{mv_1^2}{2} + \frac{kx_1^2}{2}$$

] Specific equation for this system

$$\Rightarrow K_2 + U_2 = K_1 + U_1$$

] Conservation of energy for system governed by conservative forces

here: we see $\frac{mv^2}{2} + \frac{kx^2}{2}$ = some constant

(determined by initial conditions or problem set-up)

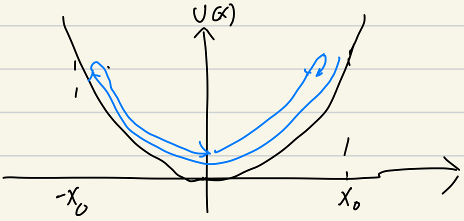
Example: A spring is stretched by x_0 from its equilibrium position and released from rest ($v_0 = 0$). What is its speed when block is at position x?

Conservation of mechanical energy:

$$M\cancelto{0}{\frac{v_0^2}{2}} + \frac{kx_0^2}{2} = \frac{mv^2}{2} + \frac{kx^2}{2}$$

$$\Rightarrow \frac{mv^2}{2} = \frac{k}{2}(x_0^2 - x^2)$$

$$\Rightarrow v = \sqrt{\frac{k}{m}(x_0^2 - x^2)}$$



Another conservative Force: gravity!

$U_g(y) = mgy$ ("gravitational potential energy")

$F_g = -\frac{dU_g}{dy} = -mg$

$$W_g = \int_1^2 \vec{F}_g \cdot d\vec{r} = -mg\int_1^2 dy = \overbrace{mg(y_1-y_2)}^{\text{again, indep. of path}}$$

⇒ conservative

Again: $\underbrace{K_2+U_2}_{} = K_1+U_1$

"mechanical energy": sum of a system's total KE (all moving parts) and total PE (all sources of potential energy)

Mechanical energy is conserved (constant), unless non-conservative Forces act on it.

Lecture 9: Energy, part 2

[Logistics: · practice exam on canvas]
[· Last HW pre-exam]

Recap of last class:

"Work-energy Theorem":

$$\Delta KE = W = \int (\Sigma \vec{F})\cdot d\vec{r}$$

"Conservative forces": (1) independent of path
(2) can be derived from a potential energy

$$F(x) = \frac{-dU}{dx}$$ ← (version we'll use, not -∇E)

So: if work is done by a conservative force,

$$\Delta KE = K_2 - K_1 = \int_{x_1}^{x_2} F(x)\,dx = -\int_{x_1}^{x_2} \frac{dU}{dx}\,dx$$

$$= U(x_1) - U(x_2)$$

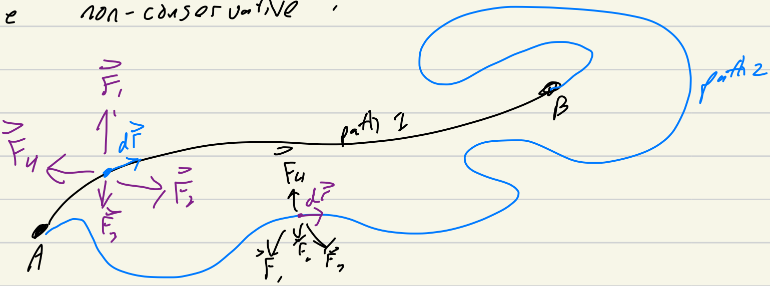
"1" and "2" label arbitrary configurations of system

Conservation of mechanical energy: $E_{mech} = K_2 + U(x_2) = K_1 + U(x_1)$

i.e.: $E_{mech} = \frac{m\vec{v}^2}{2} + U(x)$ is conserved; i.e. doesn't change over time.

Watermelon / pendulum demo.

But: Work-energy theorem holds even if forces are non-conservative:



$$W = \int_A^B \left(\sum \vec{F}\right)\cdot d\vec{r} = \Delta KE$$

Forces along specific path

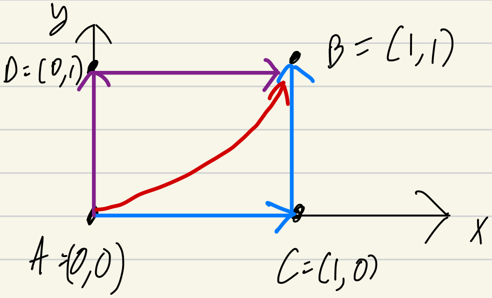
integral along specific path

In general: answer depends on path from A to B.

Example: Suppose there's a general force like

$$\vec{F}(x,y) = xy\,\hat{i} + x^3y^2\,\hat{j}$$

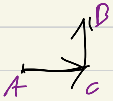
and it pushes an object from $A = (0,0)$ to $B = (1,1)$ along one of three paths



How much work is done by this force?

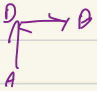
→

$W = \int \vec{F}\cdot d\vec{r}$; $\vec{F}(x) = xy\,\hat{i} + x^3y^2\,\hat{j}$



Path 1 : $W_{ACB} = W_{AC} + W_{CB}$

$$= \underbrace{\int_0^1 F_x(x, y=0)\,dx}_{0} + \int_0^1 F_y(x=1, y)\,dy = \int_0^1 y^2\,dy = \frac{1}{3}\text{ J}$$



Path 2 : $W_{ADB} = W_{AD} + W_{DB}$

$$= \int_0^1 F_y(x=0, y)\,dy + \int_0^1 F_x(x, y=1)\,dx = \int_0^1 x\,dx = \frac{1}{2}\text{ J}$$

Path 3 : along $y = x^2$ path ?

$$W = \int F_x(x,y)\,dx + F_y(x,y)\,dy$$ .. How to make progress?

Along path, $y = x^2$, and $dy = \frac{dy}{dx}dx = 2x\,dx$

$$\Rightarrow W = \int_0^1 F_x(x, y=x^2)\,dx + F_y(x, y=x^2)\cdot 2x\,dx$$

$$= \int_0^1 \left(x\cdot x^2 + (x^3\cdot x^4)\cdot 2x\right)dx = \int_0^1 x^7 + 2x^8\,dx$$

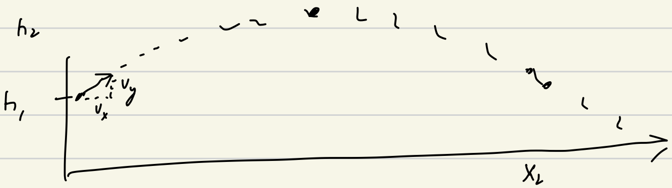
$$= \left[\frac{x^4}{4} + \frac{2x^9}{9}\right]_0^1$$

$$= \frac{1}{4} + \frac{2}{9} = \frac{17}{36}\text{ J}$$

Conservative Force example: gravity:

$$\vec{F}_g = -mg\hat{j} \quad ; \quad U_g(y) = mgy.$$

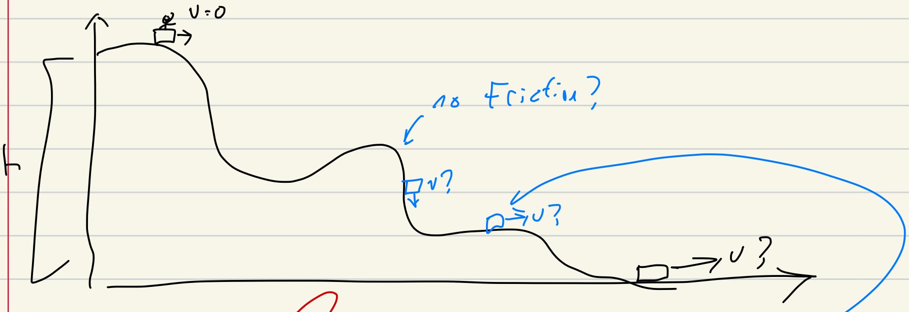
$$E_{mech} = K_1 + U_g(y_1) = K_2 + U_g(y_2)$$



height? $\frac{mv_x^2}{2} + \frac{mv_y^2}{2} + U(h_1) = \frac{mv_x^2}{2} + 0 + U(h_2)$

$$\Rightarrow \cancel{m}gh_2 - \cancel{m}gh_1 = \frac{\cancel{m}v_y^2}{2}$$

$$\Rightarrow (h_2 - h_1) = \frac{v_y^2}{2g}$$



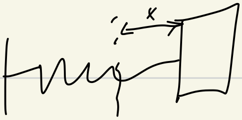
$$\frac{1}{2}\cancel{mv_i^2} + mgh = \frac{1}{2}mv^2 + 0$$

$$\Rightarrow v^2 = \sqrt{2gh}$$

Note really difference in heights!

What about normal Force??

$\Delta KE = \int(\vec{F}_g + \vec{F}_N)\cdot d\vec{r}$, and $\vec{F}_N \cdot d\vec{r} = 0$ everywhere springs:



$$F_s(x) = -kx, \quad U_s(x) = \frac{kx^2}{2}$$

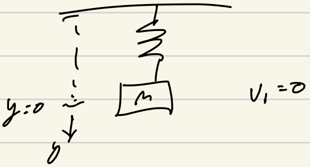
"integrate a derivative" is why we get path indep!

$$W_s = \int_{x_1}^{x_2} F_s(x)\,dx = -\int_{x_1}^{x_2} \frac{dU_s}{dx}\,dx = U_s(x_1) - U_s(x_2)$$

Notice: Forces control motion, and we always see difference of potential energies...
⇒ adding a constant to your definition of PE doesn't change anything in a physical description of your problem

---

How far will a spring hang?



After spring stops bouncing: $\Delta KE = K_2 - K_1 = W_g + W_s$

$$\cancel{\tfrac{1}{2} m v_f^2} - \cancel{\tfrac{1}{2} m v_i^2} = mg(y_f - \cancel{y_i}) + \left(\tfrac{1}{2} K y_f^2 - \cancel{\tfrac{1}{2} K y_i^2}\right)$$

$$0 = mg y_f + \tfrac{1}{2} K y_f^2$$

$$-mg \cdot \frac{2}{K} = y_F$$

Lecture 10: Energy! (Part 3)

Review of energy concepts so far:

(1) An object of mass m and velocity $\vec{v}$ has Kinetic energy

$$KE = \frac{1}{2} m v^2 \quad \left(v^2 = |\vec{v}|^2 = \vec{v} \cdot \vec{v} \right)$$

(2) Work energy theorem:

$$W = \Delta KE = K_f - K_i ,$$

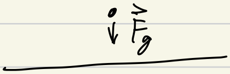
where W is total work done by all forces:

$$W = \int_i^f (\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n) \cdot d\vec{r} = W_1 + W_2 + \ldots W_n$$

(3) Conservative Forces: those for which work done by force does not depend on trajectory (only on initial and final configurations)

For conservative forces, there is an associated potential energy

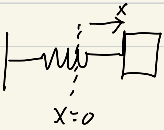
ex 1: gravitational force



$$\vec{F}_g = -mg\hat{j} = -\frac{d}{dy}(U_g(y))\,\hat{j} ;$$

where $U_g(y) = mgy$

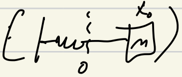
Ex 2: spring (elastic) force



$$\vec{F}_s = -kx\,\hat{i} = -\frac{d}{dx}(U_s(x))\,\hat{i}$$

where $U_s(x) = \frac{1}{2} k x^2$

(4) For conservative Forces:
E_{mech} (defined as $E_{mech} = KE + U$) is conserved.

Suppose a mass on a spring is pulled to x_0 (

) then released:

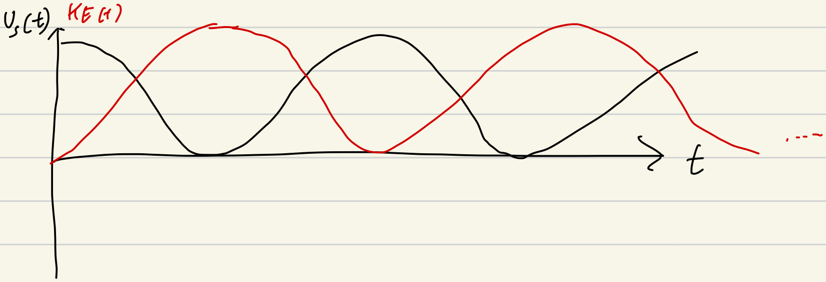
$$E_{mech} = KE + U \text{ is always the same}$$

$$E_{mech,i} = 0 + k\, x_0^2/2 \, , \quad E_{mech,later} = \frac{1}{2} m v^2 + \frac{k x^2}{2}$$

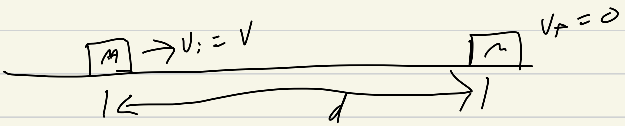
$$\Rightarrow \frac{m v^2}{2} = \frac{k}{2}\left(x_0^2 - x^2\right)$$

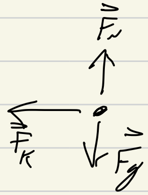
In words: as KE ↑, U_s ↓ (i.e., block is closer to $x = 0$)
as KE ↓, U_s ↑ (i.e., block is closer to $\pm x_0$)

Here: system starts at max U_s, then loses U_s and gains KE until V_{max} at $x = 0$, then loses KE until $x = -x_0$, where U_s is max again, . . .



Non-conservative effects: Friction!





$E_{mech,i} = \frac{mv_i^2}{2}$; $E_{mech,f} = 0$

Work-energy theorem: $W = \Delta KE$

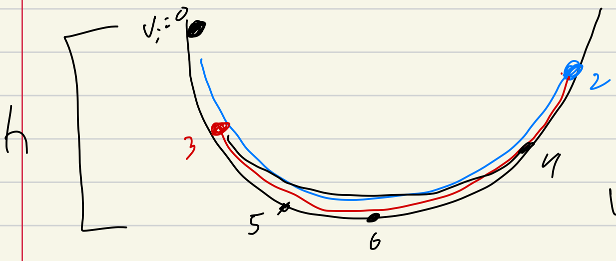
$\Rightarrow W_g + W_N + W_f = \Delta KE$, and $W_g = W_N = 0$!

So: $W_f = \Delta KE = 0 - \frac{m}{2}v_i^2$

i.e.: $W_f = \Delta KE < 0$!

Consistent with the definition of $W = \vec{F}\cdot d\vec{r}$, and for friction $\vec{F}$ points opposite to $d\vec{r}$, always.

Suppose you have a block sliding in a half-pipe:



← Sequence of $v=0$ positions.

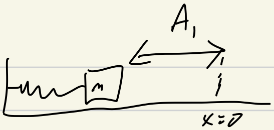
$W_{friction/cycle} = \Delta E_{mech} < 0$,

So $E_{mech,1} > E_{mech,2} > \ldots\ldots$

$E_{mech,1} = \cancel{\frac{mv_i^2}{2}} + mgh$; $E_{mech,f} = 0$, so

$W_{friction,\,total} = \Delta E_{mech} = 0 - mgh$

Spring with Friction ?



?

Well, $\Delta KE = W = W_s + W_F$

$$\Rightarrow KE_2 - KE_1 = \underbrace{\int_{x_1}^{x_2} F_s(x)\,dx}_{} + W_F$$

$\int F_s = -\int \frac{dU}{dx} = -U$

$$\Rightarrow \left(\frac{mv_2^2}{2} + U_s(x_2)\right) - \left(\frac{mv_1^2}{2} + U_s(x_1)\right) = \underbrace{W_F}_{}$$

in general, depends on path, and $W_F < 0$.

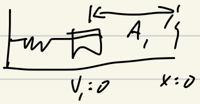
Physical Fact: loss of mech. energy is compensated by an increase in "internal" energy. In this case, the object heats up, and

$$\Delta E_{mech} + \underbrace{\Delta E_{internal}}_{= -W_F} = 0$$

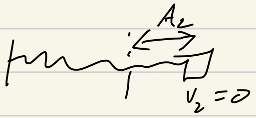
i.e.: total energy is conserved.

In this case, we know the path taken.

If we want to know



to



?

$$(KE_2 + U_2) - (KE_1 + U_1) = W_F$$

$$\Rightarrow \left(0 + \frac{k}{2}A_2^2\right) - \left(0 + \frac{k}{2}A_1^2\right) = -|F_N|\mu_K \cdot (\text{distance}) = -\mu_K mg\,(A_1 + A_2)$$

$$\Rightarrow \frac{k}{2}A_2^2 + \mu_K mg\,A_2 + \left(\frac{k}{2}A_1^2 + \mu_K mg\,A_1\right) = 0,$$

Solve for A_2 via quadratic equation.

Final concept: Power: work delivered by a Force per time

We know in a small bit of trajectory that

$$dW = \vec{F} \cdot d\vec{r}$$ (W and Energy have units of J)

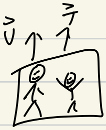
$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$

SI units: $\frac{J}{s} = W$ (= Watt)

$$1W = 1\frac{J}{s} = 1\frac{kg\,m^2}{s^3}$$

Examples: Free Fall: $\vec{v} \downarrow \downarrow \vec{F}_g$; $P_g = \vec{F}_g \cdot \vec{v} = mg\vec{v}$

lifting an elevator at constant $\vec{v}$?



$\downarrow F_g = M_{total}\, g$

$$\sum \vec{F}_y = m a_y = 0$$

$$\Rightarrow T - M_{total}\, g = 0$$

$$P_{rope} = T \cdot v = M_{total} \cdot g \cdot v$$

Phys 151 | Lecture 11 | exam review | 10/3/22

Key concepts (Following formula sheet)

$$\begin{tikzcd} \vec{r} \arrow[r, bend left, "d/dt"] & \vec{v} \arrow[l, bend left, "\int dt"] \arrow[r, bend left, "d/dt"] & \vec{a} \arrow[l, bend left, "\int dt"] \end{tikzcd}$$

$$\frac{dt^n}{dt} = n t^{n-1}$$

$\Sigma \vec{F} = m\vec{a}$; $\vec{F}_{ab} = -\vec{F}_{ba}$

$|F_{s,max}| = \mu_s |F_N|$; $|F_k| = \mu_k |F_N|$

$KE = \frac{1}{2} m v^2$

$W^{1\to2} = \int_1^2 \vec{F} \cdot d\vec{r}$

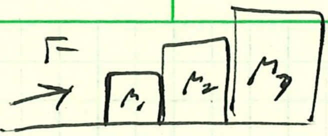
$\Delta KE = W$

$\mathcal{E}_{mech} = KE + U$; conserved if only conservative forces

conserv. force $\Longleftrightarrow$ potential $-\frac{dU}{dx} = F(x)$

[example of novice/expert problem a la UIUC?]

Q1 !



, no Friction.

A) $\vec{F}_{net} = m\vec{a}$, and $\vec{a}$ is the same for all blocks

$\Rightarrow$ block 3 has largest net Force

B) $\vec{F}_{push} = (m_1+m_2+m_3)a$

$F_{23} = F_{net\ on\ 3} = m_3 a$

$F_{21} = -F_{12}$; and $F_{12} + F_{32} = F_{net\ on\ 2} = m_2 a$

$F_{12} - m_3 a = m_2 a \Rightarrow F_{12} = (m_2+m_3)a$

$F_{2,net} = m_2 a$

(C) const. $\vec{v} \Rightarrow a = 0$

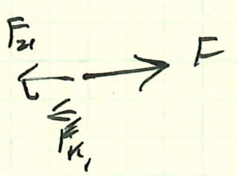
$\vec{F}_{push} = |F_{K,total}| = \mu_K (m_1+m_2+m_3) g$

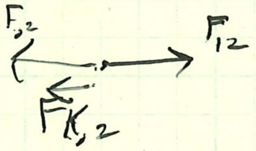
~~$F_{23} = |F_{net\ on\ 3}|$~~ $= F_{23} - F_{K3} = 0 \Rightarrow F_{23} = \mu_K m_3 g$

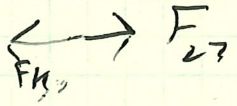
$F - F_{21} - F_{K,1} = 0 \Rightarrow F_{21} = \mu_K(m_1+m_2+m_3)g - \mu_K m_1 g$

So: $\vec{F}_{push}$ still biggest ; but all blocks have same net Force.

block 1 block 2 block 3



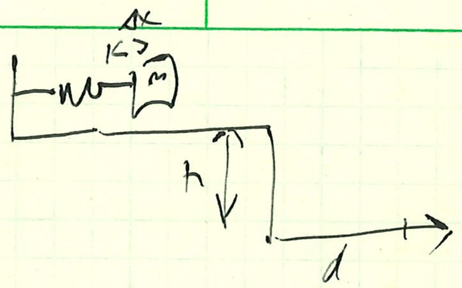




$F_{23} = \mu_K m_3 g$

$F = \mu_K(m_1+m_2+m_3)g$ $F_{12} = \mu_K(m_2+m_3)g$

Q2



(A) $\mathcal{E}_i = \frac{1}{2}k(\Delta x)^2$, $\mathcal{E}_f = \frac{1}{2}mv^2$

$\Rightarrow v = \sqrt{\frac{k}{m}}\,\Delta x$, and $\vec{v}$ points along $\hat{i}$

(B) indep. of y- and x motion:

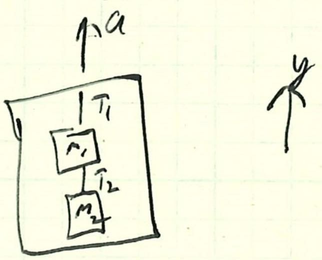
$V_x = \Delta x\sqrt{\frac{k}{m}}$ is constant.

time to fall? $h = \frac{1}{2}gt^2$ $(v_{y,i}=0,\ a=-g$

$\Rightarrow t = \sqrt{\frac{2h}{g}}$

$\Rightarrow d = \Delta x\sqrt{\frac{2hk}{gm}}$

Q3



(A) Block 1

$T_1 - T_2 - m_1 g = m_1 a$

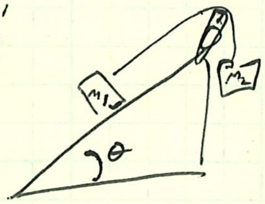
Block 2

$T_2 - m_2 g = m_2 a$

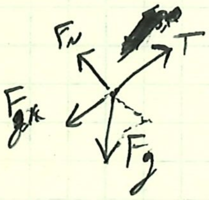
(B) $T_2 = m_2 (g+a)$

$T_1 = m_1 (g+a) + m_2 (g+a)$

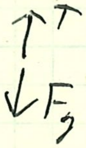
Q4:



block 1



block 2



(A) if $a = 0$:

$T = m_2 g$, and $T - F_s - F_{gx} = m_2 g - \mu_s m_1 g\cos\theta - m_1 g\sin\theta$

~~$T - F_s = m_2 g - \mu_s m_1 g\cos\theta - m_1 g\sin\theta = 0$~~

~~$\Rightarrow m_1 = \frac{[illegible]}{\mu_s\cos\theta}$~~

$$\Rightarrow m_1 = \frac{m_2}{\mu_s\cos\theta + \sin\theta}$$

(B) if m_1 slightly smaller:

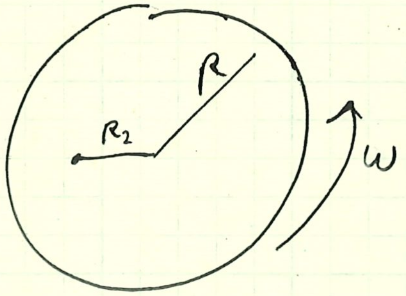
$T - m_2 g = -m_2 a \qquad \Rightarrow T = m_2(g-a)$

and

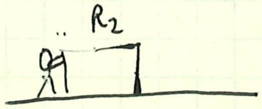
$T - F_k - F_{gx} = m_1 a \qquad \Rightarrow m_1 a = m_2(g-a) - \mu_k(\cos\theta)m_1 g - m_1 g\sin\theta$

$$\cancel{(m_1+m_2)}\,a = \frac{m_2 g - m_1 g(\sin\theta + \mu_k\cos\theta)}{m_1+m_2}$$

Q5



(A)



$|F_{mgr\ on\ child}| = m|\vec{a}_c| = m_{child} \cdot w^2 R_2$, towards center,

So $|F_{child\ on\ mgr}| = m_c w^2 R_2$, away from center

(B)

$\vec{V} = R_2 w$

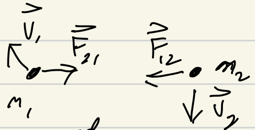
Lecture 12 : Linear momentum part 1

10/12/22

[exam return + comments]

Demo: medicine ball on a skateboard; Forces aren't easily calculable, but is there another approach we can take ??

So: consider 2 isolated particles (no external forces)



Newton's 3rd law :

$$\vec{F}_{21} = -\vec{F}_{12} \Rightarrow \vec{F}_{21} + \vec{F}_{12} = 0$$

(add 2nd law in) $\Rightarrow m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}_2}{dt} = 0$

(if masses don't change) $\Rightarrow \frac{d}{dt}\left[m_1\vec{v}_1 + m_2\vec{v}_2\right] = 0$

$$\Rightarrow \boxed{m_1\vec{v}_1 + m_2\vec{v}_2 = \text{Constant!}}$$

Does this work with more isolated particles?

$$m_1 \frac{d}{dt}\vec{v}_1 = \vec{F}_{21} + \vec{F}_{31}$$
$$m_2 \frac{d}{dt}\vec{v}_2 = \vec{F}_{12} + \vec{F}_{32}$$
$$+ \quad m_3 \frac{d}{dt}\vec{v}_3 = \vec{F}_{13} + \vec{F}_{23}$$

$\frac{d}{dt}\left[m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}\right] = 0 \Rightarrow m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 = 0$

General: For an isolated system, $\sum_i m_i\vec{v}_i$ is conserved

Define Linear Momentum : $\vec{p} = m\vec{v}$ [it's a vector: p_x, p_y, p_z]

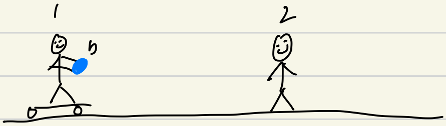
For each individual particle: $\Sigma F = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d\vec{p}}{dt}$

For the whole system: $\Sigma \vec{F}_{ext} = \frac{d}{dt}\left[\vec{p}_1 + \vec{p}_2 + \dots\right] = \frac{d}{dt}\vec{p}_{total}$

a vector equation : if, e.g., $\Sigma F_{ext,x} = 0$, then $p_{total,x}$ is constant.

Analyze skateboard example:

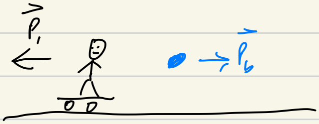
initial (skateboard removes friction for thrower)



$\vec{p}_{total} = m_1\vec{v}_1 + m_b\vec{v}_b + m_2\vec{v}_2 = 0$

---

just after throw (ball still in flight)



$\vec{p}_{total}$ = constant = 0

$\Rightarrow m_1\vec{v}_1 + m_b\vec{v}_b = 0$

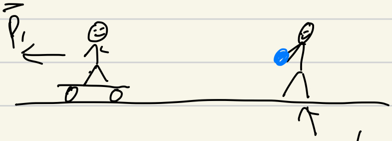
$$\boxed{\Rightarrow \vec{v}_1 = -\frac{m_b}{m_1}\vec{v}_b}$$

← only care about x-direction

in y-direction there are ext. forces

so, e.g., ball accelerates due to g...

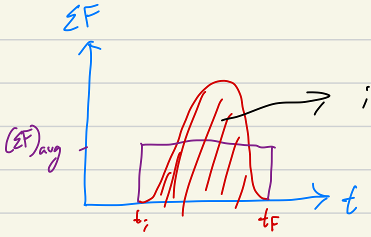
after catch



external (frictional) forces break $\vec{p}_x$ conservation; now $\vec{p}_{total} = m_1\vec{v}_1$

Impulse: If a particle is subject to forces:

$$\Sigma \vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \Delta \vec{p} = \vec{p}_f - \vec{p}_i = \underbrace{\int_{t_i}^{t_f} (\Sigma \vec{F})\, dt}_{\text{define this as } \vec{I}\text{, the impulse}}$$

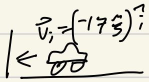


impulse imparted to particle is area under curve

$$\vec{I} = \int_{t_i}^{t_f} (\Sigma \vec{F})\, dt = \underbrace{\left(\frac{1}{\Delta t} \int_{t_i}^{t_f} (\Sigma \vec{F})\, dt\right)}_{(\Sigma F)_{avg}} \Delta t ,$$

and $\vec{I}$ tells you how much $\vec{p}$ changes

For instance: Say a car crashes in a duration of $\Delta t = 0.15s$, initially going at $17 \frac{m}{s}$ ($\approx 38 \frac{mi}{hr}$)



$m = 1500kg$

$$\vec{I} = \Delta \vec{p} = \vec{p}_f - \vec{p}_i = 0 - (1500kg)(-17\tfrac{m}{s})\hat{i} = 25500 \left(kg \tfrac{m}{s}\right)\hat{i}$$

$$\Rightarrow (\Sigma \vec{F})_{avg} = \frac{\vec{I}}{\Delta t} = (1.7 \cdot 10^5 N)\hat{i}$$ was avg. Force on car

1-D collisions : 2 limiting cases

Elastic collisions : both $\vec{p}$ and KE are conserved

Suppose we know initial velocities + masses ... what are final v's?

$\Rightarrow p_i = p_f$ and $KE_i = KE_f$

So $$m_1 v_{1,i} + m_2 v_{2,i} = m_1 v_{1,f} + m_2 v_{2,f} \qquad (1)$$

$$\frac{1}{2} m_1 v_{1,i}^2 + \frac{1}{2} m_2 v_{2,i}^2 = \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2 \qquad (2)$$

$$\Rightarrow m_1 (v_{1,i}^2 - v_{1,f}^2) = m_2 (v_{2,f}^2 - v_{2,i}^2)$$

$$\Rightarrow m_1 (v_{1,i} - v_{1,f})(v_{1,i} + v_{1,f}) = m_2 (v_{2,f} - v_{2,i})(v_{2,f} + v_{2,i}) \qquad (3)$$

From (1) : $$m_1 (v_{1,i} - v_{1,f}) = m_2 (v_{2,f} - v_{2,i}) \qquad (4)$$

divide (3) by (4) : $$v_{1,i} + v_{1,f} = v_{2,f} + v_{2,i} \qquad (5)$$

Combine (1) and (5) to solve for final v's

(algebra happens)

$$\Rightarrow v_{1,f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1,i} + \left(\frac{2 m_2}{m_1 + m_2}\right) v_{2,i}$$

and

$$v_{2,f} = \left(\frac{2 m_1}{m_1 + m_2}\right) v_{1,i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{2,i}$$

(limiting cases : if $m_1 = m_2$? if $m_1 \gg m_2$?)

perfectly inelastic collisions : particles stick together and move w/ common velocity after collision

(e.g.: ...)

well, $\vec{p}$ is still conserved!

$$m_1 \vec{v}_{1,i} + m_2 \vec{v}_{2i} = (m_1 + m_2) \vec{v}_F$$

$$\Rightarrow \vec{v}_F = \frac{m_1 \vec{v}_{1,i} + m_2 \vec{v}_{2,i}}{m_1 + m_2}$$

is KE conserved, here?

$$KE_F = \frac{1}{2}(m_1+m_2)\vec{v}_F \cdot \vec{v}_F = \frac{1}{2} \frac{1}{(m_1+m_2)} \cdot \left[m_1^2 v_{1,i}^2 + m_2^2 v_{2,i}^2 + m_1 m_2 \vec{v}_{1,i} \cdot \vec{v}_{2,i} \right]$$

$$KE_i = \frac{1}{2} m_1 \vec{v}_{1,i}^2 + \frac{1}{2} m_2 \vec{v}_{2,i}^2$$

$$KE_F - KE_i = \frac{m_1^2 v_1^2 + m_2^2 v_2^2 + m_1 m_2 \vec{v}_1 \cdot \vec{v}_2}{2(m_1+m_2)} - \frac{m_1 v_1^2 + m_2 v_2^2}{2} \cdot \frac{(m_1+m_2)}{(m_1+m_2)}$$

$$= \frac{-m_1 m_2 (v_1^2 + v_2^2 - \vec{v}_1 \cdot \vec{v}_2)}{2(m_1+m_2)}$$

and $v_1^2 + v_2^2 - \vec{v}_1 \cdot \vec{v}_2 = v_1^2 + v_2^2 - |v_1||v_2| \cos\theta$

is positive

$\Rightarrow KE_F < KE_i$

Phys151 | Lecture 13 | (Linear momentum, 2) | 10/17/22

Exam return + discussion at end.

Next H.W. posted.

Last time: $\vec{p} = m\vec{v}$ (momentum)

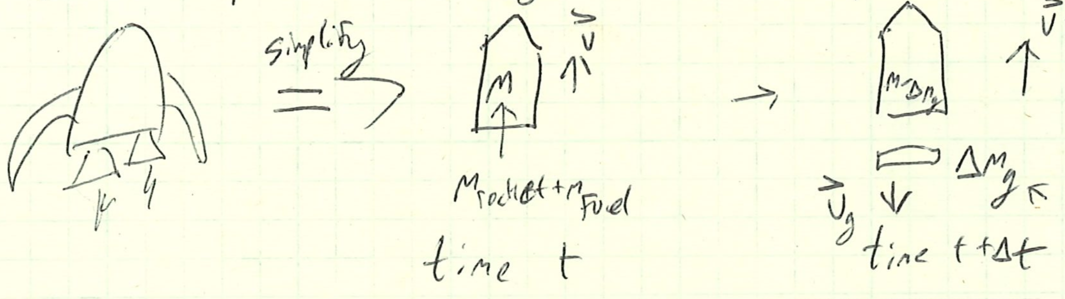
$\frac{d\vec{p}}{dt} = \vec{F} \Rightarrow \Delta\vec{p} = \vec{p}_f - \vec{p}_i = \underbrace{\int_{t_i}^{t_f} \vec{F}\, dt} \rightarrow \vec{I}$, impulse

$\vec{p}_{total} = \vec{p}_1 + \vec{p}_2 + \ldots$, and $\frac{d\vec{p}_{total}}{dt} = \Sigma \vec{F}_{ext}$

("all internal forces cancel as action-reaction pairs")

Rocket propulsion: what if mass isn't constant?

→ [Set up fire extinguisher demo... pretend it will work?]



In this set up: Fuel burns, expelling gas at constant speed, U_g, burning fuel @ rate $\frac{dm_g}{dt}$

w/ respect to rocket!

Ignore ext. forces ("gravity for rocket, friction for skateboard")

momentum conservation: (1-D)

$$p(t) = p(t+\Delta t)$$

$$\Rightarrow mv = (m - \Delta m_g)(v + \Delta v) + \Delta m_g (v - U_g)$$

$$\Rightarrow \cancel{mv} = \cancel{mv} + m\Delta v - \cancel{v\Delta m_g} - \overset{small}{\cancel{\Delta m_g \Delta v}} + \cancel{v\Delta m_g} - \Delta m_g U_g$$

$$\Rightarrow 0 = m\Delta v - U_g \Delta m_g$$

divide by Δt, take limit;

→

$$\left[\, m\frac{dv}{dt} = u_g \frac{dm_g}{dt} \Rightarrow ma = u_g \frac{dm_g}{dt} \right.$$
$$\left. \Rightarrow u_g \frac{dm_g}{dt} \text{ is the } F_{thrust} \,\right]$$

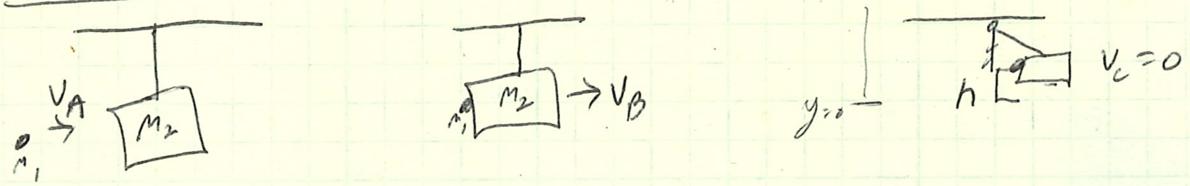
Back to our eqn: $dv = \frac{u_g\, dm_g}{m}$, and $dm_g = -dm$, so

$$dv = -u_g \frac{dm}{m}$$ … "similar to tension + friction problem"

$$\int_{v_i}^{v_f} dv = \int_{m_i}^{m_f} \left(-u_g \frac{dm}{m}\right)$$

$$\boxed{\Rightarrow v_f - v_i = u_g \ln\left(\frac{m_i}{m_f}\right)}$$

Ballistic pendulum example



how high does pendulum swing? Analyze in parts.

1) The collision is perfectly inelastic

$$\vec{P}_A = m_1 V_A \hat{i}; \quad \vec{P}_B = (m_1+m_2) V_B \hat{j}$$

$$\Rightarrow V_B = \frac{m_1}{m_1+m_2} V_A$$

2) After collision, only conservative forces, so now we can use energy conservation:

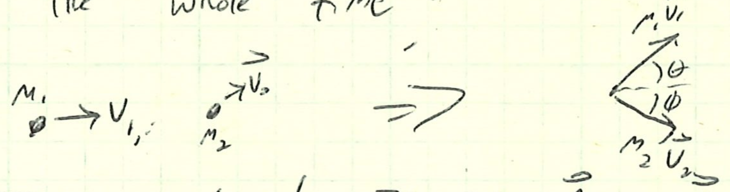
$$KE_c + \text{"}mgh\text{"} = KE_B + 0$$

$$\Rightarrow (m_1+m_2)gh = \frac{1}{2}\cdot(m_1+m_2)\cdot\left(\frac{m_1 V_A}{m_1+m_2}\right)^2$$

$$\Rightarrow h = \frac{1}{2g}\left(\frac{m_1 V_A}{m_1+m_2}\right)^2$$

units? $m = \frac{s^2}{m}\cdot\left(\frac{kg\, m/s}{kg}\right)^2 = m$ ✓

Collisions in 2D : No new concepts. It's vector equations the whole time.



IF no external Forces: $\vec{P}_i = P_F$

$$\Rightarrow m_1 V_{1i,x} + m_2 V_{2i,x} = m_1 V_{1F,x} + m_2 V_{2F,x}$$

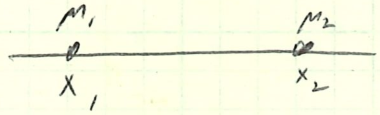
$$m_1 V_{1iy} + m_2 V_{2iy} = m_1 V_{1Fy} + m_2 V_{2Fy}$$

IF elastic, also $KE_i = KE_F$

IF perf. inelastic, $\vec{V}_{1F} = \vec{V}_{2F}$, etc.

Center of Mass:

in 1D



, let $M = m_1 + m_2$ = total mass

Newton's Laws

$$m_1 a_1 = F_{ext,1} + F_{21}$$

$$+ \quad m_2 a_2 = F_{ext,2} + F_{12}$$

$$m_1 \frac{d^2x_1}{dt^2} + m_2 \frac{d^2x_2}{dt^2} = (F_{ext,1} + F_{ext,2})$$

"will come back to that"

Define

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$V_{cm} = \frac{1}{m_1 + m_2}\left(m_1 \frac{dx_1}{dt} + m_2 \frac{dx_2}{dt}\right)$$

$$P_{cm} = M V_{cm} = m_1 \frac{dx_1}{dt} + m_2 \frac{dx_2}{dt} = P_1 + P_2$$

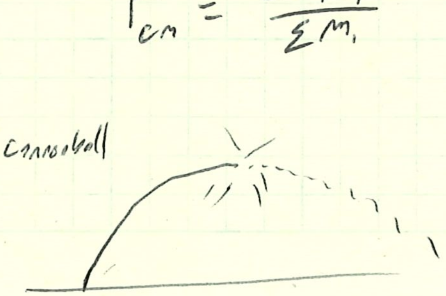
So: $P_{cm} = P_{total}$, and $\frac{dP_{cm}}{dt} = m_1 \frac{d^2x_1}{dt^2} + m_2 \frac{d^2x_2}{dt^2} = \sum F_{ext}$.

So: CM moves like a particle of mass M subject to total ext Forces.

Generalization: $x_{cm} = \frac{\sum m_i x_i}{\sum m_i}$;

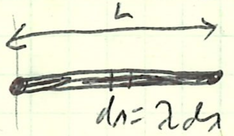
$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

exploding cannonball



CM For continuous system : Sums to integrals.

example : rod of length L, uniform mass per length, λ :



$$\frac{M}{L} = \lambda$$

$$x_{cm} = \frac{1}{M}\sum_i x_i \Delta m_i = \frac{1}{M}\int_0^L x\,dm$$

$$= \frac{1}{M}\int_0^L x\,\lambda\,dx = \frac{\lambda}{M}\left[\frac{x^2}{2}\right]_0^L = \frac{\lambda L^2}{2M} = \frac{L}{2}$$

if λ wasn't uniform? $\lambda \to \lambda(x)$, still do the integral!

Talk about exam :

Points to hit : I want everyone to succeed, and believe you all can. I'm also sure many left the exam feeling like it could have gone better.

Stats (raw vs scaled)

why scale? Time pressure, so discount lowest Q, lightly scale other Qs

How to interpret scores

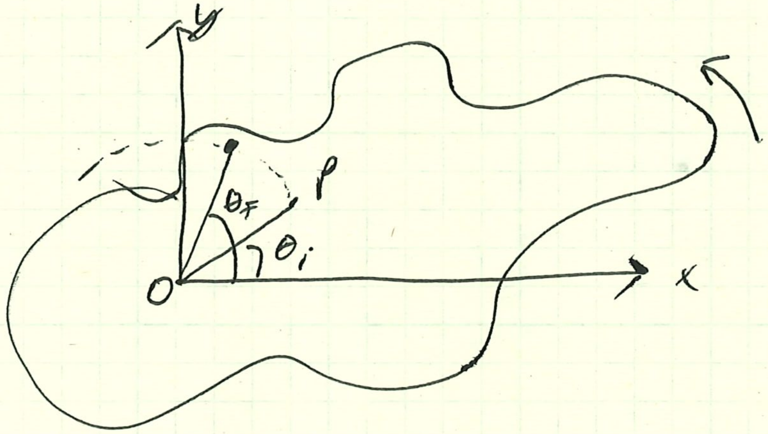
Still time → think about how to study... (remember practice exam? Make more use of office hours / Phys Mentors?

Set up a time to talk w/me...

Phys 151 | Lecture 14 | Rotations, part 1 | 10/19/22

[Exam - get help / office hours this week]

<u>Rigid body rotation</u> : An object is in x-y plane and rotates about z-axis, which passes through O.



P : <u>any</u> point in object

θ : measured CCW w.r.t x-axis

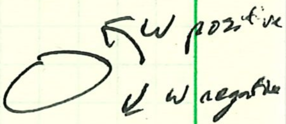
"rigid body" : every point, P, in object describes <u>circular</u> motion with fixed radius, r, relative to axis of rotation.

Just like when we studied kinematics : there is a relationship between Δs ("arc-length") and $\Delta\theta$ (measured in radians) : $\Delta s = (\Delta\theta)\cdot r$ [recall $1\,\text{rad} = \frac{360}{2\pi}^\circ$]

1) Average angular velocity : $\omega_{avg} = \frac{\Delta\theta}{\Delta t}$ $\xrightarrow{2)}$ $\omega = \frac{d\theta}{dt}$

3) angular acceleration : $\alpha = \frac{d\omega}{dt}$

4) convention :



---

motion at constant α :

$\alpha = \text{const}$

$\omega = \omega_i + \alpha t$

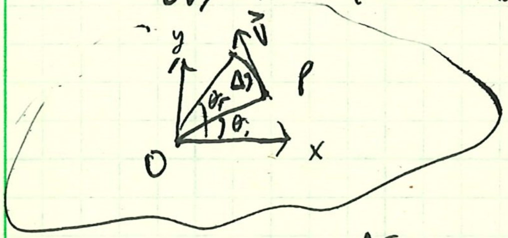
$\theta = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$

⟹ Same kinematic structure as position/vel/acc!

$$\begin{tikzcd}
\theta \arrow[r, bend left, "d/dt"] & \omega \arrow[l, bend left, "\int dt"] \arrow[r, bend left, "d/dt"] & \alpha \arrow[l, bend left, "\int dt"]
\end{tikzcd}$$

Angular vs translational variable for rotational motion

How do we talk about $\vec{v}$ $\vec{a}$ of points in a rotating object that isn't otherwise moving through space?



speed $$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{r\Delta\theta}{\Delta t} = r \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = r\omega$$

$$a_t = \frac{dv}{dt} = \frac{d(r\omega)}{dt} = r\frac{d\omega}{dt} = r\alpha$$

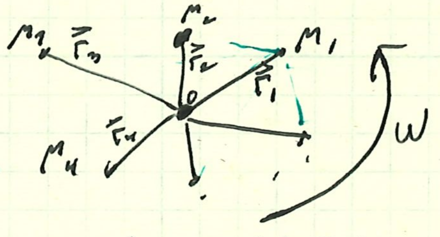
$$a_c = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$$

[we'll talk about vectors and cross-products later!]

Rotational KE:

Suppose we have a bunch of points with different masses, m_i and positions, $\vec{r}_i$, rotating around O
rigidly

will be modified on next page... draw big on board



What is KE, here? Use definition!

rigid body: same ω for all points

$$KE = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (r_i\omega)^2 = \left(\sum_i \frac{1}{2} m_i r_i^2\right)\omega^2$$

"not impulse"

Define $I = \sum_i m_i r_i^2$, "moment of inertia"

$\Rightarrow KE = \frac{1}{2} I \omega^2$ ... "looks" analogous to old KE formula.

But notice: I depends on masses and where masses are relative to the axis of rotation!

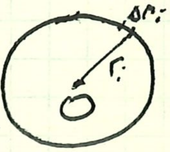
For instance: suppose O passes through m_1:

[Draw new distances, notice that $r_1 = 0$, so m_1 doesn't appear in I any more.]

What about continuous objects?

Compute I by thinking of object as a bunch of poinst of mass!

ex 1: Ring of mass M rad R, about O



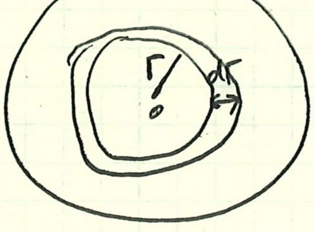
$r_i = R$ for all bits of mass

total mass $= \sum_i \Delta m_i = M$

$$\Rightarrow I = \sum_i m_i r_i^2 = \sum_i m_i R^2 = R^2 \sum_i m_i$$

$$= M R^2$$

ex 2: solid disk, uniform density, of mass M, radius R, through O:



Every different radius r is like a little ring:

$$dI = r^2 \cdot dM = r^2 \cdot \boxed{\frac{M}{\pi R^2}} \cdot 2\pi r \, dr$$

$$I_O = \int_0^R dI = \int_0^R r^2 \cdot \frac{M}{\pi R^2} \cdot 2\pi r \, dr$$

about center.

$$= \frac{2M}{R^2} \int_0^R r^3 \, dr = \frac{2M}{R^2} \left[\frac{r^4}{4} \right]_0^R$$

$$= \frac{1}{2} M R^2$$

What is analogy of Force for rotations?

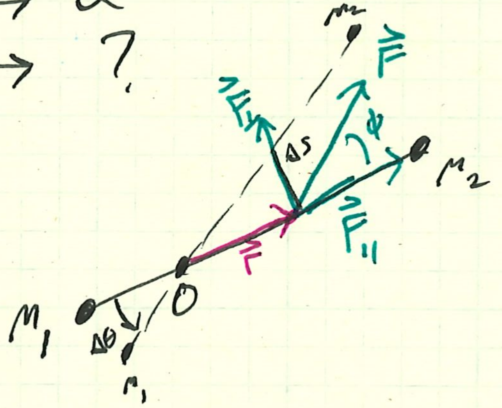
$\vec{r} \leftrightarrow \theta$ and axis of rotation

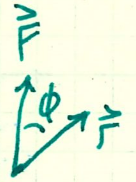
$v \leftrightarrow \omega$ " "

$a \leftrightarrow \alpha$ " "

$\vec{F} \leftrightarrow$?

Torque!





Suppose $\vec{F}$ causes a rotation of a small amount $\overset{\Delta\theta}{\widehat{\Delta s}}$ in time Δt ... what work was done?

$$dW = F_\perp ds = \underbrace{|\vec{F}|\sin\phi \cdot r}\, d\theta$$

τ, the torque...

$dW_{rot} = \tau\, d\theta$ just like $dW_{trans} = \vec{F}\cdot d\vec{r}$

[again, vector version soon]

Power: $P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega$ a la $P = \vec{F}\cdot\vec{v}$

Rotational Work-Energy Theorem for rigid body rotation.

slightly different from book derivation; same result!

$dW = d(KE)$ ← rigid-body, I isn't changing.

$\tau_{ext}\, d\theta = d\left(I\frac{\omega^2}{2}\right) = I\underbrace{\omega}\, d\omega \rightarrow \omega = \frac{d\theta}{dt}$

$\Rightarrow \tau_{ext}\, d\theta = I\, d\theta \frac{d\omega}{dt}$

$\Rightarrow \tau_{ext} = I\frac{d\omega}{dt}$

$= I\alpha$

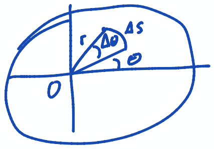
So: we see τ causes α!

Lecture 15 : Rotational Motion, part 2.

[HW 9 posted]

Review :

$$\begin{tikzcd} \theta \arrow[r, bend left, "d/dt"] & \omega \arrow[l, bend left, "\int dt"] \arrow[r, bend left, "d/dt"] & \alpha \arrow[l, bend left, "\int dt"] \end{tikzcd}$$

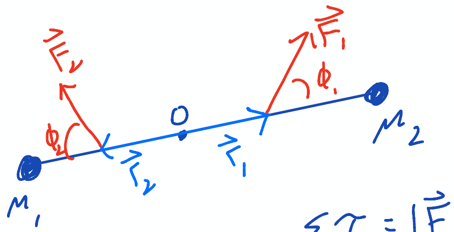


$\Delta S = r\Delta\theta$

$v = r\omega$

$a_t = \alpha r$

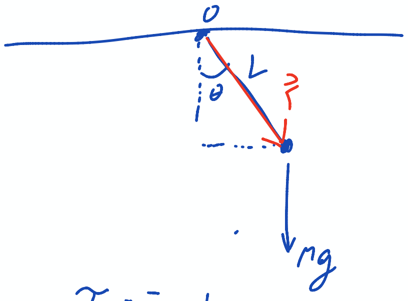
$KE_{rot} = \frac{1}{2} I \omega^2$; $\quad I = \sum_i m_i r_i^2$ [depends on axis of rotation!]



$$\sum \tau_{ext} = I\alpha$$

$$\sum \tau = \underset{\substack{\uparrow \\ \text{rotates ccw}}}{|\vec{F}_1| \cdot |\vec{r}_1| \sin\phi_1} - \underset{\substack{\uparrow \\ \text{rotates cw.}}}{|\vec{F}_2| \cdot |\vec{r}_2| \cdot \sin\phi_2}$$

Simple pendulum :



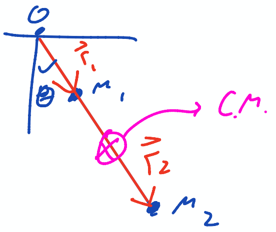
$\tau_g = -mgL\sin\theta$; $I = mL^2$

$-mgL\sin\theta = mL^2\alpha$

$\Rightarrow \alpha = \frac{-g\sin\theta}{L}$; if θ small, $\sin\theta \approx \theta$!

$$\left(\overset{\text{or}}{a = -g\sin\theta} \right)$$

2-mass pendulum:



$M_{total} = m_1 + m_2$

$$\tau_g = -m_1 g r_1 \sin\theta - m_2 g r_2 \sin\theta = -g M_{total} \cdot \left(\frac{m_1 r_1 + m_2 r_2}{M_{total}} \right) \cdot \sin\theta$$

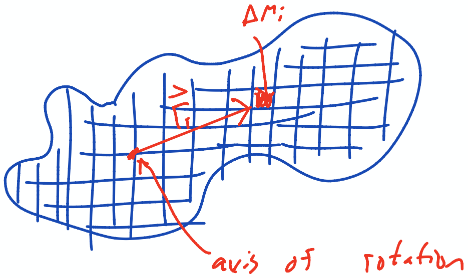
$$= -g M_t \cdot r_{cm} \sin\theta$$

$\Rightarrow$ External torque due to gravity can be calculated by thinking of the torque due to a point mass located at the C.M.!

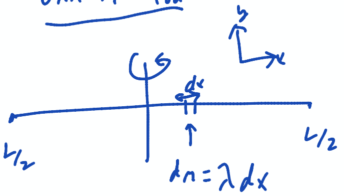
e.g.

Moment of Inertia : For continuous object:

$$I = \lim_{\Delta m \to 0} \sum_i r_i^2 \Delta m = \int r^2 dm$$



uniform rod

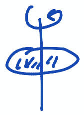


$dm = \lambda dx$, $\lambda = \frac{M}{L}$

$$\Rightarrow I = \int_{-L/2}^{L/2} x^2 dm = \int_{-L/2}^{L/2} \lambda x^2 dx$$

$$= \frac{ML^2}{12}$$

Solid disk

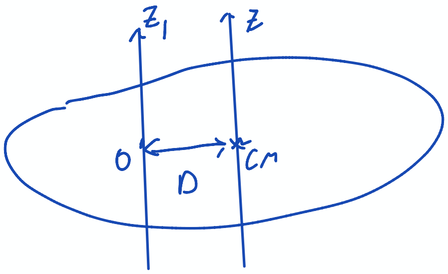


$I = \frac{MR^2}{2}$

Sphere

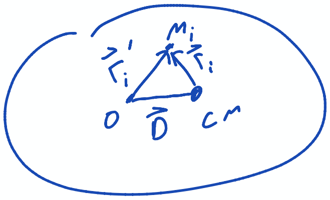
$I = \frac{2MR^2}{5}$

Parallel axis theorem :



Theorem : $I_0 = I_{cm} + MD^2$

proof :



So: $\vec{r}_i\,' = \vec{D} + \vec{r}_i$

Let's have a coordinate system where $\vec{r}_{cm} = 0$ . . .

$$I_0 = \sum_i m_i r_i'^2 = \sum_i m_i \vec{r}_i\,' \cdot \vec{r}_i\,' = \sum_i m_i (\vec{D} + \vec{r}_i) \cdot (\vec{D} + \vec{r}_i)$$

$$= \underbrace{\sum_i m_i r_i^2}_{= I_{cm}} + \underbrace{\sum_i m_i D^2}_{= MD^2} + \underbrace{2\left(\sum_i m_i \vec{r}_i\right) \cdot \vec{D}}_{\propto \vec{r}_{cm} = 0}$$

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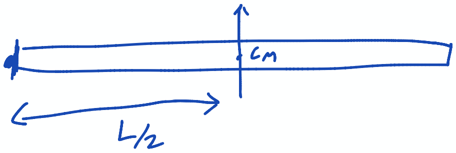
example : Rod pivoting around end



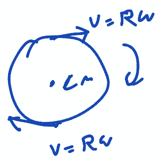
$$I = I_{cm} + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$$

(as computed differently last time)

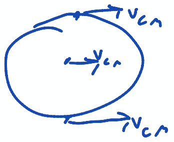
Rolling without slipping:

[rolling demo]

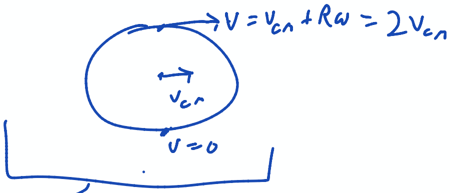
pure rotation



pure translation



rolling w/ no slip



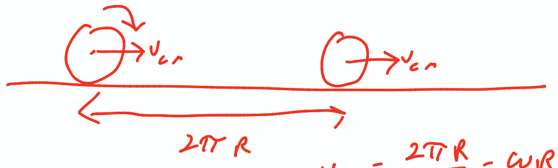
KE here is just like rotating around the ground contact point, P,

$$KE = \frac{1}{2} I_p \omega^2 = \frac{1}{2}(I_{cm} + MR^2)\omega^2$$

$$= \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M R^2 \omega^2$$, and $R\omega = V_{cm}$, so

$$KE = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M v_{cm}^2$$

Why?

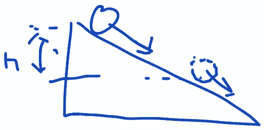


$$V_{cm} = \frac{2\pi R}{T} = \omega R$$

Can also write

$$KE = \frac{1}{2}\left(\frac{I_{cm}}{R^2} + M\right) V_{cm}^2$$

Example



no slip ⇒ conservative

$$\Delta KE = -mg\Delta h$$

$$\frac{1}{2}\left(\frac{I}{R^2} + M\right)v^2 = mgh \Rightarrow V_{cm} = \sqrt{\frac{2gh}{1 + I_{cm}/MR^2}}$$

disk: $I = \frac{1}{2} M R^2$ vs hoop $I = MR^2$

$v = \sqrt{\frac{2gh}{1+\frac{1}{2}}}$

$= \sqrt{\frac{4}{3}gh}$

$v = \sqrt{\frac{2gh}{1+1}}$

$v = \sqrt{gh}$

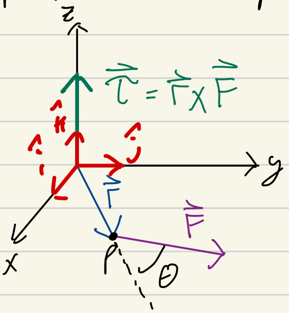
10/25/22

Lecture 16: Rotations (part 3) and static Equilibrium

[Logistical Notes:

Torques and Vector products

Suppose, as an example, $\vec{r}$ and $\vec{F}$ are in the x-y plane:



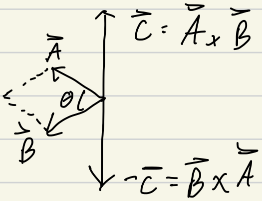
$\vec{r}$ points to P, and $\vec{F}$ acts on a mass at P

Torque is a <u>vector</u> quantity:

<u>Magnitude</u> is $|\vec{\tau}| = |\vec{r}| \cdot |\vec{F}| \sin\theta$

<u>Direction</u> is given by "right hand rule"

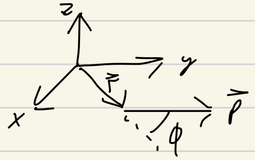
<u>Vector product</u>:



<u>Algebra</u>: $\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Angular momentum: Suppose a particle, mass m, has momentum $\vec{p}$



Define Angular momentum relative to origin as

$$\vec{L} = \vec{r} \times \vec{p}$$

[here, if $\vec{r}$ and $\vec{p}$ are in xy plane $L = rp\sin\phi\,\hat{z}$]

If there are forces:

$$\vec{F}_{net} = \frac{d}{dt}\vec{p}$$

("Newton's second law")

$$\Rightarrow \vec{\tau}_{net} = \vec{r} \times \vec{F}_{net} = \vec{r} \times \frac{d\vec{p}}{dt} = \frac{d}{dt}(\vec{r} \times p) - \underbrace{\left(\frac{d\vec{r}}{dt}\right) \times \vec{p}}_{= \frac{1}{m}\vec{p}\times\vec{p} = 0}$$

$$\Rightarrow \vec{\tau}_{net} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d}{dt}\vec{L}$$

* "Just like forces cause changes in linear momentum, torques cause changes in angular momentum"

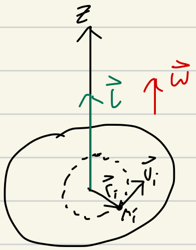
* Important: both $\vec{\tau}$ and $\vec{L}$ are calculated relative to a point!

For a collection of particles:

$$\vec{L}_{total} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \dots = \sum_i \vec{L}_i$$

and $$\sum \vec{\tau}_{ext} = \frac{d}{dt}\vec{L}_{total}$$

Angular momentum of rotating object:



Same as $\hat{K}$

$$\vec{L}_i = m_i v_i r_i \overbrace{\sin\frac{\pi}{2}}^{1}\,\hat{z} = m_i r_i^2 \omega\,\hat{z}$$

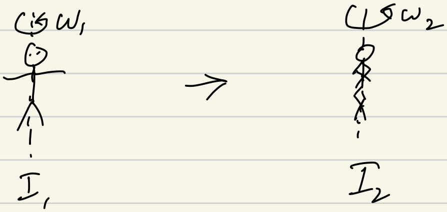
$$\vec{L} = \sum_i \vec{L}_i = \left(\sum_i m_i r_i^2\right)\omega\,\hat{z}$$

$$\Rightarrow \boxed{\vec{L} = I\vec{\omega}}$$

Angular momentum is conserved when $\sum \vec{\tau}_{ext} = 0$!

$$\sum \vec{\tau} = 0 = \frac{d}{dt}\vec{L} \Rightarrow \vec{L} \text{ const, i.e., } \vec{L}_i = \vec{L}_f$$

Classic example: Figure skating spin



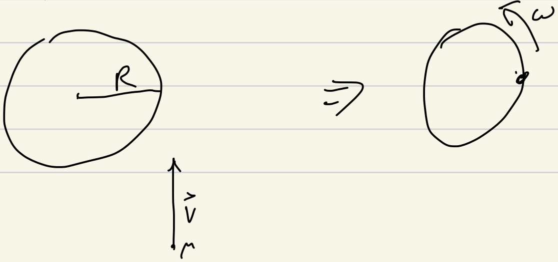
, $I_1 > I_2$
(think about where mass is)

Ignore friction ⇒ no ext. torques, so

$$I_1\vec{\omega}_1 = I_2\vec{\omega}_2$$

$$\Rightarrow \omega_2 = \frac{I_1}{I_2}\omega_1$$

Collision w/ disk:



$L_i = mvR$

$L_f = (I_{disk} + mR^2)\omega$

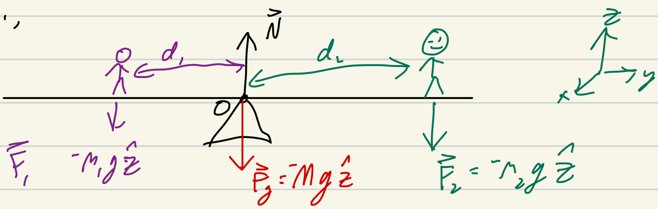
$$\Rightarrow \omega = \frac{mvR}{I_{disk} + mR^2}$$

Static Equilibrium

"Static Eq." means

$$(1)\ \sum \vec{F} = 0 \qquad \text{"translational Equilibrium"}$$
$$(2)\ \sum \vec{\tau} = 0 \qquad \text{"rotational equilibrium"}$$

Ex 1 : "see-saw"



IF in static eq. : $\sum \vec{F} = 0$

$$\Rightarrow N - (m_1 + m_2 + M)g = 0$$
$$\Rightarrow \vec{N} = (m_1 + m_2 + M)g\ \hat{z}$$

and $\sum \vec{\tau} = 0$

$$\Rightarrow \vec{\tau}_1 + \vec{\tau}_2 + \cancel{\vec{\tau}_g} + \cancel{\vec{\tau}_N} = 0$$

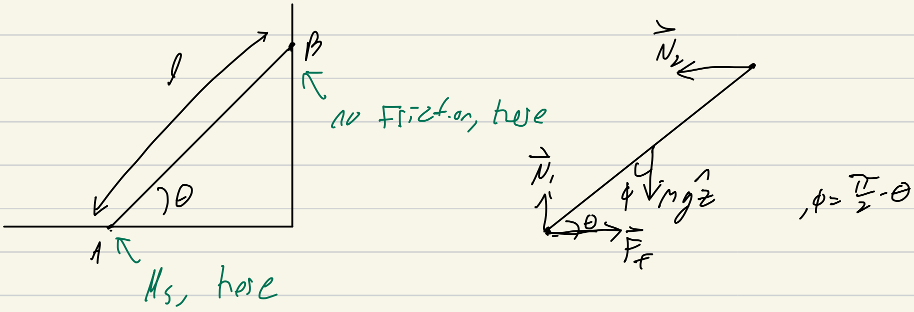
$\hat{x} \times \hat{z} =$

$$(-d_1 \hat{x}) \times (-m_1 g)\hat{z} + (d_2 \hat{x}) \times (-m_2 g \hat{z}) = 0$$
$$-d_1 m_1 g\ \hat{y} + d_2 m_2 g\ \hat{y} = 0$$
$$\Rightarrow m_1 d_1 = m_2 d_2 \quad \text{or} \quad \frac{m_1}{m_2} = \frac{d_2}{m_1}$$

exercise : the board isn't moving, so it isn't rotating about any axis .. show $\sum \tau$ about person 1 is also zero ?

Another example

A ladder is against a frictionless wall.. how much can it lean?



$\Sigma \vec{F} = 0 \Rightarrow \Sigma F_y = 0 \Rightarrow N_1 = mg$

$\Sigma F_x = 0 \Rightarrow F_f = N_2$

how big can F_f be? $F_{f,max} = \mu_s N_1 = \mu_s mg$

$\Sigma \tau$ about A?

$$\Sigma \tau = 0$$

$$N_2 \, l \sin\theta - mg \frac{l}{2} \underbrace{\sin\left(\frac{\pi}{2} - \theta\right)}_{= \cos\theta} = 0$$

$$\Rightarrow (\mu_s mg) \, l \sin\theta - mg \frac{l}{2} \cos\theta = 0$$

$$\Rightarrow \tan\theta = \frac{1}{2\mu_s}$$

tells us the angle we need!

10/31/22

Phys 151

Lecture 17: Universal Gravitation

[Logistics : · this week
· next week

- All objects with mass attract each other.

Newton: The same gravitational force causes that attraction, from apples falling to Earth to the Earth "falling" towards the sun!

$$F_g \propto M_1 M_2 \; F(r)$$

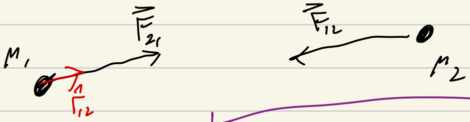
Object falling near Earth's surface:

$$F_g = \cancel{m} M_E \, F(R_E) = \cancel{m} a \quad , \quad a = M_E \, F(R_E)$$

Earth falling towards sun:

$$F_g = \cancel{M_E} M_S \, F(r_{ES}) = \cancel{M_E} a_E \; ; \; a_E = M_E \, F(r_{ES})$$

What is this function of distance?



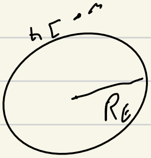
$$\vec{F}_{12} = -\vec{F}_{21} ,$$

just expressed as prop. to inverse square of dist — Newton ~ 1687

$$\boxed{\begin{aligned} \vec{F}_{12} &= \frac{-G M_1 M_2}{r^2} \hat{r}_{12} \\ G &= 6.674 \cdot 10^{-11} \, \frac{N m^2}{kg^2} \end{aligned}}$$

First good measurement of G? Cavendish, 1798

Free-Fall near earth:



$$|F_g| = \frac{G m M_E}{r^2}$$

Calculate as if all mass of earth is concentrated at Earth's center!

$$= G m M_E \cdot \frac{1}{(R_E + h)^2}$$

$$\text{So "}g\text{"} = \frac{G M_E}{(R_E+h)^2} \approx \frac{G M_E}{R_E^2} \quad \text{if } h \ll R_E$$

$$\approx 9.8 \, m/s^2$$

Aside: From this, estimate

$$\rho_E = \frac{3 M_E}{4\pi R_E^3} = \frac{3}{4\pi} \frac{g}{G R_E} \approx 5.5 \cdot 10^3 \, kg/m^3$$

Newton's Canon and orbiting objects

neglecting air resistance, orbiting is just constantly falling towards an object at the same rate that the surface falls away!

$$m \, a_c = \frac{G m M_E}{(R_E + h)^2} = m \frac{v^2}{(R_E + h)}$$

$$\Rightarrow v = \sqrt{\frac{G M_E}{R_E + h}}$$

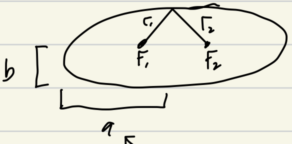
Near surface, $v = \sqrt{\frac{G M_E}{R_E}} \approx 7.8 \, km/s$

Keplers Laws:

From extensive observation of planets and stars in night sky (mostly by Tycho Brahe, late 1500's) Kepler developed a model that reproduced data (1609–1619)

1st Law: Planets move in ellipses, w/ Sun at focus

Ellipse:

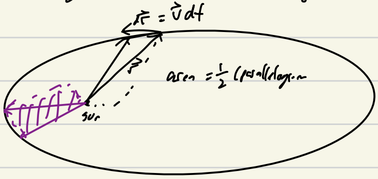


$r_1 + r_2 = \text{const}$

"Semi-major axis"

Circle is special case of $F_1 = F_2$

2nd Law: A vector From sun to planet sweeps out equal areas in equal time intervals:



M = mass of planet

Angular momentum: $\vec{L} = \vec{r} \times (M\vec{v})$

$= M\,(\vec{r} \times \vec{v})$

torque From gravity: $\vec{\tau} = \vec{r} \times \vec{F} = 0$,

So $\vec{L}$ is constant

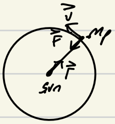
So: $dA = \frac{1}{2}\,|\vec{r} \times d\vec{r}| = \frac{1}{2}\,|\vec{r} \times \vec{v}|\,dt$

$$\frac{dA}{dt} = \frac{1}{2M}\,|\vec{r} \times (M\vec{v})| = \frac{|L|}{2M} = \text{constant}$$

Note: only requirement for this result was (a) isolated system, (b) central Force

3rd Law: "$T^2 \propto a^3$"

Simple version: suppose orbit is a circle



$$m_p a = F_g = \frac{G M_p M_E}{r^2} = M_p \left(\frac{v^2}{r}\right)$$

orbital speed $v = \frac{2\pi r}{T}$

$$\Rightarrow \frac{G M_p M_E}{r^2} = M_p \cdot \frac{1}{r} \cdot \frac{4\pi^2 r^2}{T^2}$$

$$\Rightarrow T^2 = r^3 \cdot \frac{4\pi^2}{G M_s}$$

For ellipses, eventually get same result For semi-major axis

$$T^2 = a^2 + \frac{4\pi^2}{G M_S}$$

Grav. Potential Energy

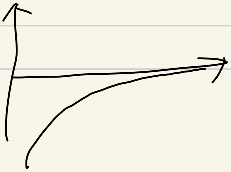
we have $F_g(r) = -\frac{G M_1 m_2}{r^2}$

and Know $F(r) = -\frac{dU}{dr}$ ... what is U_g?

$$\Delta U = U_f - U_i = -\int_{r_i}^{r_f} F_g(r)\,dr = \left[\int_{r_i}^{r_f} \frac{1}{r^2}\,dr\right] \cdot G M_1 m_2$$

$$= G M_1 m_2 \left[-\frac{1}{r}\right]_{r_i}^{r_f}$$ ... define $r_i = \infty$, $U_i = 0$,

$$\Rightarrow U_g(r) = -\frac{G m_1 m_2}{r}$$



how does this relate to our earlier expression?

$$U_g(R_E+h) = \frac{-GMm}{R_E(1+h/R_E)}$$

$$\approx -\frac{GMm}{R_E}\left(1-\frac{h}{R_E}\right) = -\frac{GMm}{R_E} + \frac{GMm}{R_E^2}h$$

So: $U_g(R_E+h) - U_g(R_E) = m\left(\frac{GM_E}{R_E^2}\right)h$

$\nwarrow$ = "g = 9.8 $\frac{m}{s^2}$" !

Energy of a bound orbit / escape velocities

For a circular orbit:

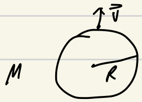
$$\mathcal{E} = KE + U_g = \frac{1}{2}mv^2 + \left(-\frac{GMm}{r}\right)$$

and $m\left(\frac{v^2}{r}\right) = \frac{GMm}{r^2}$ From a_c

$$\Rightarrow \mathcal{E} = \frac{m}{2}\left(\frac{GM}{r}\right) - \frac{GMm}{r} = -\frac{GMm}{2r}$$

($\mathcal{E}$ is negative because $U_g(r\to\infty)=0$)

Escape velocity:



Use energy balance

$$\frac{1}{2}mv_e^2 - \frac{GMm}{R} = \cancelto{0}{\frac{1}{2}mv_f^2} - \cancelto{0}{\frac{GMm}{r_f}}$$

$$\Rightarrow v_{escape} = \sqrt{\frac{2GM}{r}}$$

For earth: $v_{escape} \simeq 11$ km/s

Phys151 | Lecture "19" | Midterm 2 review | 11/9/22

Concept review:

Linear momentum: $\vec{p} = m\vec{v}$; conservation: $\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}$

Impulse: $\vec{I} = \Delta\vec{p} = \int_{t_1}^{t_2} \vec{F}_{net}\, dt$

perfectly elastic collision: momentum and KE conserved

inelastic collision: just momentum is conserved

Center of mass: $\vec{r}_{cm} = \frac{1}{M_{total}} \cdot \sum_i m_i \vec{r}_i$

Angular Kinematics: $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$; $\vec{\omega} = \frac{d\vec{\theta}}{dt}$

Cross product: $\vec{A} \times \vec{B} = |A| \cdot |B| \sin\theta$, points in direction of RHR.

Torque about point: $\vec{\tau} = \vec{r} \times \vec{F}$ ($\vec{r}$ is from point to where force acts)

Angular momentum about axis:

$$\vec{L} = \vec{r} \times \vec{p}$$

For rotating about a fixed axis: $\vec{L} = I\vec{\omega}$

Moment of Inertia: $I = \sum_i m_i r_i^2 \rightarrow \int r^2 dm$

parallel axis theorem: $I = I_{cm} + MD^2$ (D is dist from cm to axis of rotation)

examples: $I_{cm, rod} = \frac{1}{12}ML^2$; $I_{cm, disk} = \frac{1}{2}MR^2$; $I_{cm, sphere} = \frac{2}{5}MR^2$

Conservation of angular momentum: $\vec{\tau} = \frac{d\vec{L}}{dt}$

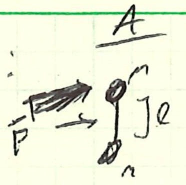
Gravity: $\vec{F}_{12} = \frac{-Gm_1m_2}{r^2}\hat{r}_{12}$; $U_g(r) = \frac{-Gm_1m_2}{r}$ (recall, $F(x) = -\frac{dU}{dx}$)

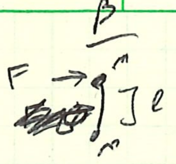
Kepler: 1. orbits are ellipses

2. equal areas in equal times

3. $T^2 = \left(\frac{4\pi^2}{GM}\right)a^3$ (For circular orbits, this is just circular motion, centrip. accel., and F_g)

Q1:





(A) which gets more CM speed?

Same: $\vec{F}_{ext} = M_{total}\,\vec{a}_{cm}$, so CM motion is same

(B) In which case does F do more work?

Case B. use work-energy theorem; both have same V_{cm}, but B also has rotational KE

(C) For case A, if masses are doubled, but same force is applied for the same time, how does $\vec{p}_{Final}$ change?

Stays the same! Impulse: $\Delta p = F\Delta t \Rightarrow \vec{p}_f$ is the same

Q2

(elastic collision

what is μ_K?

(i) how fast is m_2 going when it hits m_1?

energy conservation: $m_2 g l = \frac{1}{2} m_2 v^2 \Rightarrow v_{2i} = \sqrt{2gl}$

(same answer if use $KE = \frac{1}{2} I \omega^2$

(ii) how fast does m_1 go after collision?

elastic! $m_2 v_{2,i} + m_1 v_{1,i} = m_2 v_{2F} + m_1 v_{1F}$ and

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1F}^2 + \frac{1}{2} m_2 v_{2F}^2$$

From the KE one (plus algebra) get

$v_{2F} - v_{1F} = v_{1i} - v_{2i}$, and $v_{1i} = 0$

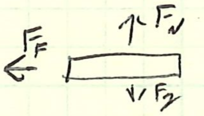
$\Rightarrow v_{2F} = v_{1F} - v_{2i}$

plug into momentum:

$$m_2 v_{2i} = m_2 (v_{1F} - v_{2i}) + m_1 v_{1F}$$

$$\Rightarrow \frac{2 m_2 v_{2i}}{m_1 + m_2} = v_{1F}$$ (and $v_2 = \sqrt{2gl}$

(iii) Find μ_K:



$F_F = -\mu_K m g$

Use work-energy: Friction is only non-cons. Force, so it's responsible for change in KE:

$$\frac{1}{2} m_1 \cancel{v}_{1F}^2 = \vec{F}_F \cdot \vec{d} = \mu_K m_1 g d$$

$$\Rightarrow \mu_K = \frac{1}{2gd} \left(\frac{2 m_2 v_{2i}}{m_1 + m_2} \right)^2 = \frac{2}{dg} \left(\frac{m_2}{m_1 + m_2} \right)^2 \cdot 2gl$$

$$= 4 \left(\frac{l}{d} \right) \left(\frac{m_2}{m_1 + m_2} \right)^2$$ $\leftarrow$ Unitless

(can also use kinematics to relate a, F_F, d, t, etc)

~~[illegible]~~
$dm = \lambda dx$

Q3: Rod w/ length L, linear mass density $\lambda(x) = A + B(x - \frac{L}{2})^2$, For $0 \le x \le L$

(A) total mass $M = \int_0^L \lambda(x)\,dx = \int_0^L A + B(x - \frac{L}{2})^2 dx$

$$= AL + \int_0^L Bx^2 - BLx + \frac{BL^2}{4}\,dx$$

$$= AL + B\left[\frac{x^3}{3} - \frac{Lx^2}{2} + \frac{L^2x}{4}\right]_0^L = AL + BL^3\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{4}\right]$$

$$= AL + B\frac{L^3}{12}$$

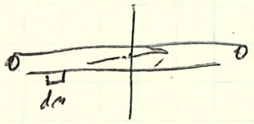
(B) where is CM? Can calculate $x_{cm} = \frac{1}{M_{total}} \cdot \int_0^L x\lambda(x)\,dx$

$= \_\ \_\ \_\ \_\ \_\ \_$

Or: notice

mass is symmetric about $L/2$, so $x_{cm} = L/2$

(C) I_{cm} ?



let x be dist from cm!

[redacted]

$$I = \int r^2\,dm = \int_{-L/2}^{L/2} r^2(A + Br^2)\,dr$$

$$= \ldots = \frac{AL^3}{12} + \frac{BL^5}{80}$$

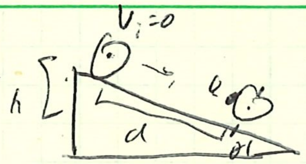
Alternate: $I_0 = I_{cm} + M_{total} D^2$; $I_0 = \int_0^L x^2(A + B(x - \frac{L}{2})^2)\,dx$

$$= \frac{AL^3}{3} + \frac{BL^5}{30}$$

$$I_{cm} = I_0 - (AL + B\frac{L^3}{12}) \cdot (\frac{L}{2})^2$$

$= \ldots = \frac{AL^3}{12} + \frac{BL^5}{80}$ … same answer.

Q4



disk rolls w/ out slipping

dist. d down a ramp.

(A) CM speed of disk ?

$$I_{cm\,disk} = \frac{1}{2} M R^2$$

$$E_{mech,i} = M g d \sin\theta$$ (no KE, choose height of cm at bottom of ramp as $y = 0$)

$$E_{mech,F} = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2$$

and $$\omega = \frac{V_{cm}}{R}$$

$$\Rightarrow M g d \sin\theta = \frac{1}{2} M v^2 + \frac{1}{2}\left(\frac{M}{2} R^2\right) \cdot \frac{V_{cm}^2}{R^2}$$

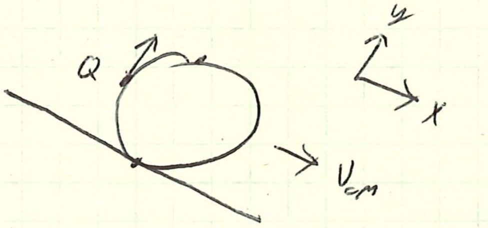
$$g d \sin\theta = V_{cm}^2 \cdot \left(\frac{1}{2} + \frac{1}{4}\right)$$

$$\Rightarrow V_{cm}^2 = \frac{4}{3} g d \sin\theta \qquad \Rightarrow V_{cm} = \sqrt{\frac{4 g d \sin\theta}{3}}$$

$$\uparrow$$

$$\sqrt{\frac{4}{3} g d \sin\theta}$$

(B) speed of point Q ?



circular motion:

$$V_{point\ on\ edge} = R\omega = V_{cm} \ !$$

$$\vec{V}_Q = V_{cm}\,\hat{i} + V_{cm}\,\hat{j}$$

$$\Rightarrow |V_Q| = \sqrt{V_{cm}^2 + V_{cm}^2} = V_{cm}\sqrt{2}$$

(5) A satellite of mass M orbits a planet of mass M_p w/ period T.

(A) Mech energy of system?

Let r be the distance from cm of planet to the satellite...

From Kepler (or derive from circular motion + gravity)

$$\frac{T^2}{r^3} = \frac{4\pi^2}{GM_p} \Rightarrow r = \left(\frac{GM_p T^2}{4\pi^2}\right)^{1/3}$$

$$U_g = -\frac{GMM_p}{r}$$

$KE = \frac{1}{2}Mv^2$ what is v?

Use centrip. accel due to gravity

$$\frac{Mv^2}{r} = \frac{GM_pM}{r^2}$$

$$\Rightarrow \frac{1}{2}Mv^2 = \frac{GM_pM}{2r}$$

$$E_{mech} = U_g + KE = -\frac{GM_pM}{r} = -GM_pM \cdot \left(\frac{4\pi^2}{GM_pT^2}\right)^{1/3}$$

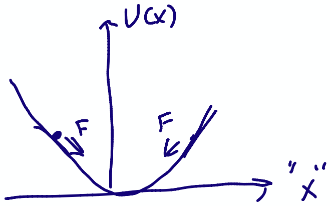
(B) to just escape: $E_{mech,f} = 0$, and $E_{mech,i} =$ ⤴, so positive that much.

Lecture 20: Oscillations

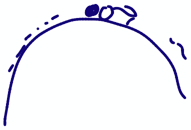
- HW on this chapter
- Office hours this week: just exams

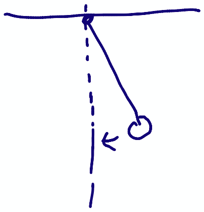
Oscillations: very general phenomenon when systems are perturbed about some stable equilibrium state.

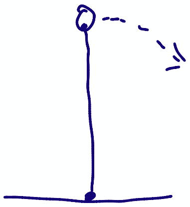
Stable eq



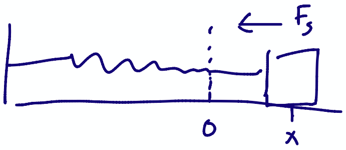
unstable eq







Harmonic Oscillator: Consider a spring-mass system:



$F_s = -Kx$

Units of K? Kg/s^2

Newton: $F(x) = ma$

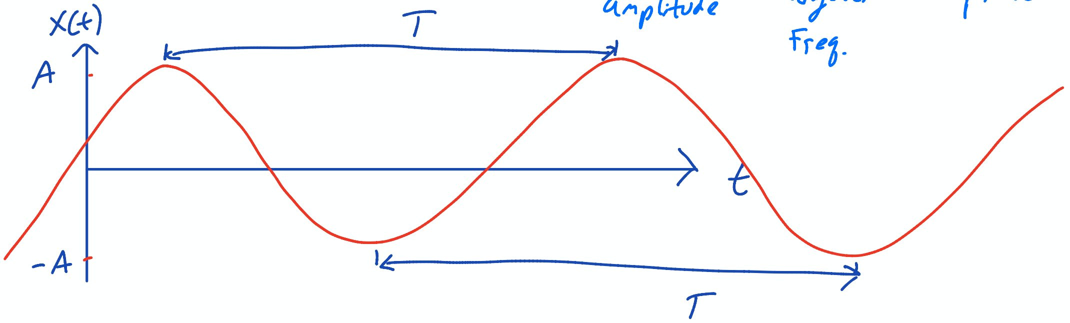
$\Rightarrow -Kx = m\dfrac{d^2x}{dt^2}$ … define $\omega = \sqrt{\dfrac{K}{m}}$ Units? $\frac{1}{s}$

$\Rightarrow \dfrac{d^2x}{dt^2} = -\omega^2 x$ … a differential equation describing simple harmonic motion

What kinds of functions solve this? sin and cos!

General solution :

$$\boxed{x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) = \underset{\text{amplitude}}{A} \cos(\underset{\text{angular Freq.}}{\omega} t + \underset{\text{"phase"}}{\phi})}$$



A : Amplitude (block oscillates between ±A)

ϕ : phase (shift origin of t)

T : period of oscillation

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

[f : Freq. of oscillations ; $f = \frac{1}{T}$ units of $\frac{1}{s}$, Hz

$$= \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Kinematics :

$$x(t) = A \cos(\omega t + \phi)$$

$$v(t) = -\omega A \sin(\omega t + \phi) \quad ; \quad v_{max} = \omega A = \sqrt{\frac{k}{m}}\, A$$

$$a(t) = -\omega^2 A \cos(\omega t + \phi) = -\omega^2 x \; ; \quad a_{max} = \omega^2 A$$

Energy : Springs are conservative : $U_s(x) = \frac{k x^2}{2}$

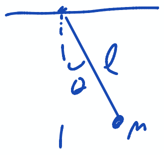
So : $\mathcal{E} = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$

$$= \frac{1}{2} m \left[\omega^2 A^2 \sin^2(\omega t + \phi)\right] + \frac{1}{2} k \left[A^2 \cos^2(\omega t + \phi)\right] \; ; \quad \omega^2 = \frac{k}{m}$$

$$= \frac{1}{2} k A^2 \left[\sin^2 + \cos^2\right]$$

$$\Rightarrow \boxed{\mathcal{E} = \frac{1}{2} k A^2} \; , \quad \text{constant!}$$

Simple pendulum :



$$\tau = -mgl\sin\theta = I\alpha$$

$$\Rightarrow -mgl\sin\theta = I\frac{d^2\theta}{dt^2} \quad ; \quad I = ml^2$$

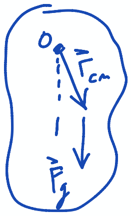
$$\Rightarrow \frac{d^2\theta}{dt^2} = \frac{-g}{l}\sin\theta$$; and for small θ, $\sin\theta \approx \theta$

$$\Rightarrow \frac{d^2\theta}{dt^2} = \frac{-g}{l}\theta$$

mathematically same as above!

$$\omega^2 = \frac{g}{l} \quad ; \quad T = 2\pi\sqrt{\frac{l}{g}}$$

Hanging blob pendulum



$\tau_0 = -mgd\sin\theta$, d is dist from pivot to cm

$$\tau_0 = I_0\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I_0}\right)\sin\theta \approx -\left(\frac{mgd}{I_0}\right)\theta$$ if θ is small

$$\omega^2 = \frac{mgd}{I_0} \quad ; \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I_0}{mgd}}$$

Damped oscillations :

put a simple harmonic oscillator in, eg, a viscous environment...

$$m\frac{d^2x}{dt^2} = -Kx - b\frac{dx}{dt}$$ ... "b" = "damping coef."

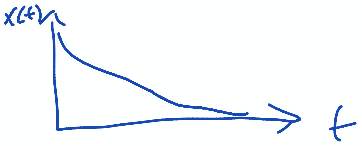
$$\Rightarrow \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega_0^2 x = 0$$; call "$\frac{b}{m}$" $= \gamma$, $\omega_0 = \sqrt{\frac{K}{m}} \equiv$ "natural frequency"

guess a possible Form For the solution

$$x(t) = e^{at}$$

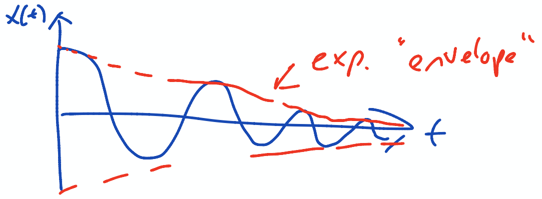
$$\Rightarrow \left[a^2 + \gamma a + w_0\right] e^{at} = 0 \quad \Rightarrow \quad a = \frac{-\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - w_0^2}$$

Case 1: $\frac{\gamma}{2} > w_0$, real roots : "overdamped oscillation"



Case 2: $\frac{\gamma}{2} < w_0$, imaginary roots

$$\Rightarrow x(t) = A e^{-\left(\frac{\gamma}{2}\right)t} \cos\left(t\sqrt{w_0^2 - \left(\frac{\gamma}{2}\right)^2} + \phi\right)$$



Forced oscillations

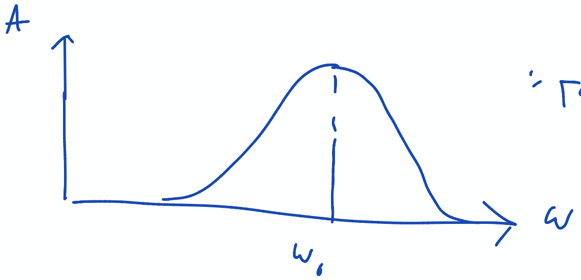
$$m\frac{d^2x}{dt} = -Kx - b\frac{dx}{dt} + \underbrace{F_0 \sin(w_F t)}_{\text{periodic Forcing, as an example}}$$

here w_F is Forcing Freq.

What happens? eventually, settle into sync w/ driving

$$x(t) = A\cos(w_F t + \phi),$$ where

$$A = \frac{F_0}{m} \cdot \frac{1}{\sqrt{(w^2 - w_0^2)^2 + \left(\frac{bw}{m}\right)^2}} \; ; \quad w_0 = \sqrt{\frac{K}{m}} \, ,$$ still "natural Freq."



"resonance" !

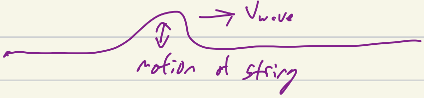
11/21/22

Lecture 21 : Waves part 1 (ch 16)

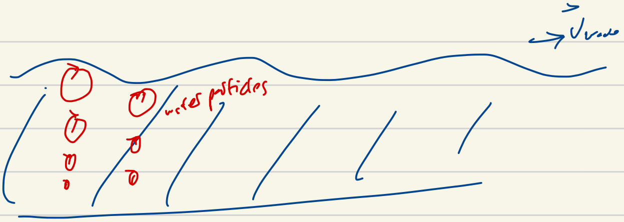
- No HW this week
- Happy Thanksgiving!

Waves : In essence, a way of propagating energy without propagating matter.

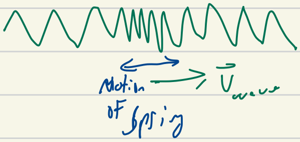
Wave on a string



wave on water



compression of springs:

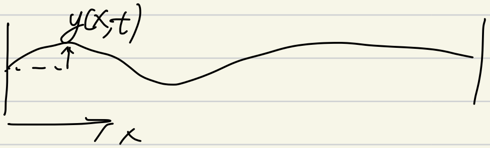


Transverse wave ; motion of medium is perpendicular to direction of propagation

longitudinal wave ; motion of medium is parallel to direction of propagation

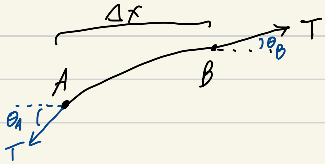
In this class, there will always be a medium... stay tuned for EM waves in the spring!

Wave on a string



We'll focus on small amplitude oscillations...

Consider a small bit of string segment under tension:
[similar to friction/tension example]



Small amplitude ⇒ θ_A and θ_B are small ⇒ $\sin\theta \simeq \tan\theta \simeq \theta$

Force in y-direction:

$$\Sigma F_y = T(\sin\theta_B - \sin\theta_A)$$

$$\simeq T[\tan\theta_B - \tan\theta_A]$$

$$= T\left[\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right]$$

Newton's second law:

$$(\Delta m)a_y = \Sigma F_y$$; let $\Delta m = \mu \Delta x$, μ is linear mass density

$$\Rightarrow \mu \Delta x \frac{\partial^2 y}{\partial t^2} = T\left[\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right]$$

$$\Rightarrow \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A}{\Delta x}$$

take limit $\Delta x \to 0$:

$$\boxed{\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \quad v = \sqrt{\frac{T}{\mu}}}$$

"The wave equation!" where v is speed of propagation.

Solutions to the wave equation

one solution: $y(x,t) = A \sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$

λ: "wavelength" : $y(x+\lambda,t) = y(x,t)$

T: "period" : $y(x,t+T) = y(x,t)$

$$\frac{vT}{\lambda} = 1 \Rightarrow v = \frac{\lambda}{T} = \lambda F,$$

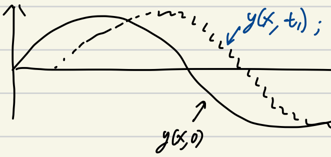
F the "Frequency"

Common to write the solution as

$$y(x,t) = A \sin\left(\frac{2\pi}{\lambda}(x - vt)\right) = A \sin\left(2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right)$$
$$= A \sin(Kx - \omega t)$$

$K = \frac{2\pi}{\lambda}$, "wavenumber"

$\omega = \frac{2\pi}{T} = 2\pi F$, "angular Frequency"



"$x - vt$" is a wave traveling to the right.

Power transmitted by waves:

See discussion in text for details ... by dimensional analysis:

$$P = \frac{\text{energy}}{\text{time}} = \frac{\text{energy}}{\text{length}} \cdot \text{speed}$$
$$= \frac{1}{2}\mu\omega^2 A^2 v$$

Intensity of sound

$I = \frac{P}{A} = \frac{P}{4\pi r^2}$ For a sound expanding in 3D

$= \frac{\text{energy}}{\text{time}} \cdot \frac{1}{\text{area}}$

Sound intensity is measured in dB

$\beta = 10 \log\left(\frac{I}{I_0}\right)$; $I_0 = 10^{-12} \frac{W}{m^2}$ "threshold of hearing"

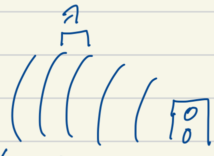
if $I = I_0$; $\beta = 0$

if $I = 1 \frac{W}{m^2}$; $\beta = 10 \log\left(\frac{1}{10^{-12}}\right) = 120$ "decibels"

(ouch!)

Doppler effect: Change in observed freq. due to relative motion of source and observer.

Observer moving



observer sees waves moving at $v' = v + v_0$, so

$$F' = \frac{v'}{\lambda} = \frac{v + v_0}{\lambda} = \frac{v + v_0}{v/F} = \left(\frac{v + v_0}{v}\right)F$$

so $F' > F$ if V_0 is towards the source

Source moving

observer measures a different wave than the source is emitting:

$$\lambda' = \lambda - \Delta\lambda = \lambda - \frac{v_s}{F}$$

$$F' = \frac{v}{\lambda'} = \frac{v}{\lambda - v_s/F} = \frac{v}{v/F - v_s/F} = \left(\frac{v}{v - v_s}\right)F$$

so; $F' > F$ when source moves towards observer.

11/28/22

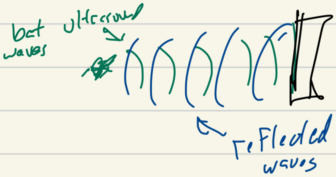
Lecture 22 ; Wave motion part 2

[• Exam comments
• Course evals → complete on Canvas]

Doppler : $F' = \left(\frac{v + v_0}{v - v_s}\right) F$

We talked about how that teaches us about car sounds as they zoom past... But much cooler!

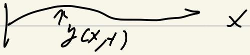
Suppose you're a bat!



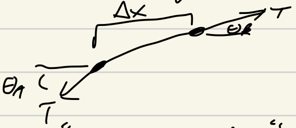
timing of reflected waves : distance to wall

Doppler shift → how fast bat is moving and prey!

Waves on a string :



By thinking about a segment of string



We derived the "wave equation" :

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \quad \dots \quad v = \sqrt{\frac{\text{Tension}}{\text{linear mass density}}}$$

And we noticed that

$y(x,t) = A \sin(Kx \pm \omega t + \phi)$ were possible solutions

$$K = \frac{2\pi}{\lambda} \quad ; \quad \omega = \frac{2\pi}{T} = 2\pi F$$

v is set by string, λ and F related as $v = \lambda F$

Goal for today: Build from there to a plucked string.

Superposition: Suppose $y_1(x,t) = A_1 \sin(k_1 x + \omega_1 t + \phi_1)$
and $y_2(x,t) = A_2 \sin(k_2 x + \omega_2 t + \phi_2)$

"Are each solutions to the wave equation. We can make another valid solution by forming any linear combination:"

$$y_3(x,t) = C_1 y_1(x,t) + C_2 y_2(x,t)$$

(C_1 and C_2 are any constants)

"So: when two waves propagate, the resulting wave is the sum of each."

[Demo of Constructive + Destructive waves on Slinky ... "peak+peak = bigger peak ... peak+trough = 0"

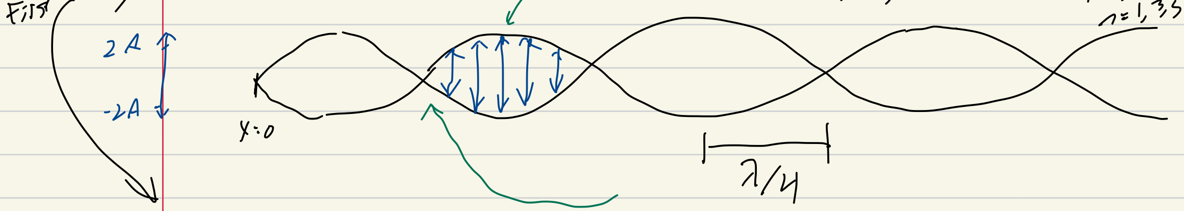
Standing waves: Suppose $y_1 = A\sin(kx - \omega t)$ and $y_2 = A\sin(\cancel{k}x + \omega t)$
"(two waves w/ same λ, f but traveling in opposite directions)"

trig identity: $\sin a + \sin b = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$

$$\Rightarrow y = y_1 + y_2 = 2A\cos(\omega t)\sin(kx)$$

This is a "standing wave," which has a stationary outline

"anti-nodes"; $kx = \frac{2\pi x}{\lambda} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \Rightarrow x = \frac{n\lambda}{4}$, $n = 1, 3, 5$

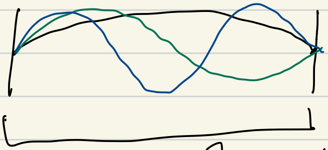


"nodes"; $kx = \frac{2\pi x}{\lambda} = 0, \pi, 2\pi, 3\pi, \ldots \Rightarrow x = n\frac{\lambda}{2}$

$n = 0, 1, 2, \ldots$

"How can we form a standing wave? one way: Clamp string at one end and send waves in: each wave will reflect off of the clamped end and travel with same λ, f but in opposite direction. Clamp both ends and a wave initially produced will keep bouncing back and forth."

"what standing wave patterns fit with both ends clamped?"



$\lambda_1 = 2L = \frac{2L}{1}$ "Fundamental"

$\lambda_2 = L = \frac{2L}{2}$ "second harmonic"

$\lambda_3 = \frac{2L}{3}$, "third harmonic"

, etc.

$\lambda_n = \frac{2L}{n}$, $n = 1, 2, 3 \ldots$

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2L} = n \cdot \underbrace{\frac{1}{2L}\sqrt{\frac{\text{Tension}}{\text{mass density}}}}_{= \text{"}f_1\text{"}} = n f_1$$

"Beats" what if we have two waves of slightly different frequencies being added? Say, $y_1 = A\sin(kx)\cos(\omega_1 t)$

$y_2 = A\sin(kx)\cos(\omega_2 t)$

(another trig identity)

$$\Rightarrow y_3 = \underbrace{2A\sin(kx)\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)}_{\text{slowly varying amplitude}} \cos\left(\underbrace{\frac{\omega_1 + \omega_2}{2}}_{\text{average } \omega \text{ for combined wave}} t\right)$$

when $\omega_1 \approx \omega_2$

example: $f_1 = 438\,Hz$, $f_2 = 442\,Hz$ one hears $f_{avg} = 440\,Hz$ going through intensity maxima/minima at $f_{beat} = 4\,Hz$

So: A string can have a specific collection of standing waves... (any amount of $f_1, f_2, f_3, \ldots$) .. what happens when we pluck a string? A plucked string doesn't look like any harmonic at all!

Fourier's theorem : Any periodic function can be approximated by terms in its "Fourier series".

suppose a function has period T, so $y(t+T)=y(t)$.

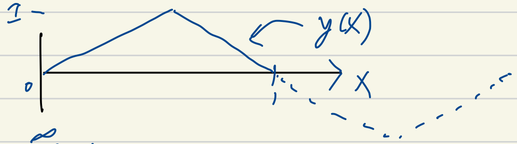
Let $F_1 = 1/T$, $F_n = n F_1$

Then

$$y(t) = \sum_{n=0}^{\infty} \left(A_n \sin(2\pi F_n t) + B_n \cos(2\pi F_n t)\right)$$

So: arbitrary waveforms (in time, or in space) can be built up by combining the "right amount" of all of the different harmonics.

Example :



$$y(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x), \quad A_n = 0 \text{ if } n \text{ is even}$$
$$A_n = \frac{4}{\pi^2 n^2} \cdot (-1)^{(n-1)/2} \text{ if } n \text{ is odd}$$

and $y(x,t) = \sum_{n=1}^{\infty} 2A_n \cos(n\pi t)\sin(n\pi x)$ is the "triangular standing wave"

$\hookrightarrow$ see animations

$\hookrightarrow$ This is why different instruments sound different: can play same fundamental freq, but it's the specific mix of harmonics that makes an instrument sound like an instrument.

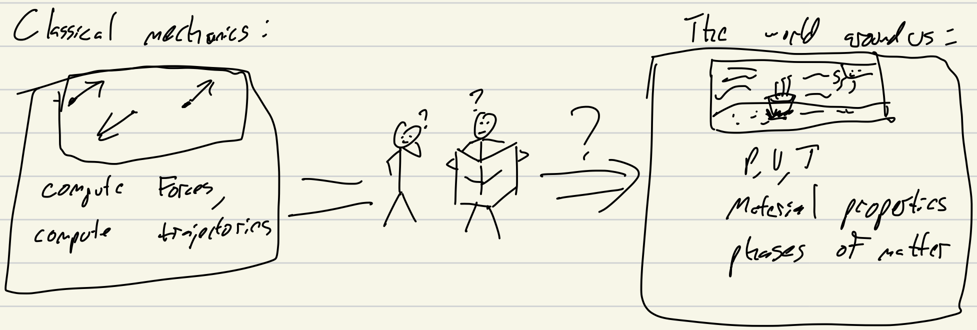
Phys 151

Lecture 23: Thermodynamics, part 1 !

[selections from ch 18 + 19]

[• Comments on Exam 2: overall performance and curve
• Structure of Final: many concept questions, a few problems, open *paper* notes (no phone/calc/computer)

Demo of tuning fork beating... for fun.



Message: Newtonian mechanics... very satisfying! I have "particles" and forces, and I can understand how things move around. But... we often want to understand the world at a very different "level of description".

e.g. Every morning I look at the pot of water for my coffee, and I want to know not how the water molecules will move around — who could possibly care about the position of N_A molecules! — but "is the water hot yet? How much heat do I need to add to get it to boil? Will the steam burn me? Why, in fact, are some molecules water vs steam? Why is water "wet," and...

How to even *begin* answering? physics is the perspective that the world is understandable, so giving up isn't an option... but neither is writing Newton's equations of motion for 10^{23} particles!

Thermodynamics is a potential answer... it developed out of things in the industrial age — what is heat, how can we design engines that do work — and evolved into a branch of physics that explains everything from that to phases of matter to what happens when you compress a file on your computer.

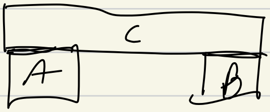
Just like we derived everything in the earlier part of the class from (a) Kinematics and (b) Newton's 3 laws, we'll look at (a) Equilibrium and (b) the laws of Thermodynamics, and build from there

"Thermal equilibrium": The basic observable properties of a blob of stuff (pressure or volume of a gas, length of a wire, the strength of a magnet) doesn't change over some period of observation.

Observation: if we put two objects of different temperatures together, they will eventually reach the same final temperature.

0th Law of Thermodynamics: "Thermal Equilibrium is transitive"

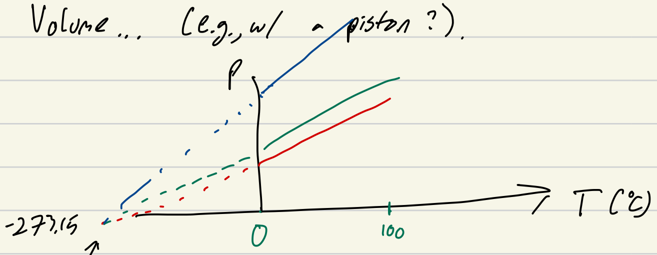
i.e. if A and B are each, separately, in T.E. with object C, then A and B are in TE w/ each other



Consequence: we can label different kinds of T.E. by a quantity we'll call "temperature"... temperature determines if energy will flow between objects placed in contact.

How to measure T? Use a thermometer!

Dilute gas thermometer: Measure P vs T at constant Volume... (e.g., w/ a piston?).



we notice that all these curves extrapolate to the same T @ zero pressure

Kelvin scale: starts at zero, and define another universal T as the triple point of ice-water-steam ($0.01\,^\circ C$ and $P = 4.58$ mm Hg)

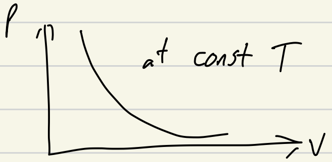
$T = 0\,K \rightarrow T = -273.15\,^\circ C$

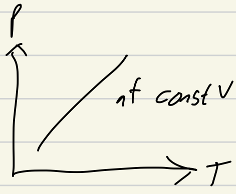
$T = 273.15\,K \rightarrow T = 0\,^\circ C$

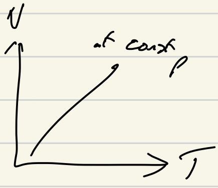
$T = 373.15\,K \rightarrow T = 100\,^\circ C$

Ideal Gas: limit of a very dilute gas, in which the interactions between molecules don't matter (because they are so rare)

Behavior:







$$\Rightarrow \quad PV = N k_B T$$

N: # of molecules

k_B: $1.38 \cdot 10^{-23} \frac{J}{K}$ (Boltzmann's const.)

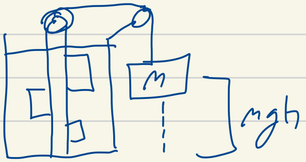
T measured in Kelvin

1st Law of thermodynamics: "Heat is a form of energy"

Not obvious! (Caloric fluid, etc) and "if you consider a stationary object, the change in that object's internal energy is equal to the heat added to the object minus the work done on the object by its environment"

$$\Delta U = Q + W$$

Joule's experiment



set up an insulated container full of water in which a falling mass could turn paddles... Joule observed that the water temperature increased!

"mgh" → heat, Q, and

$$Q = m_w \, c_w \, \Delta T = m_w c_w (T_f - T_i)$$

↑ mass of water

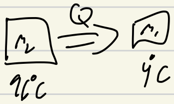
↑ "specific heat" of water, $c_w = 4.186 \frac{J}{g\,\text{Kelvin}}$

$Q > 0 \Rightarrow$ heat into system

$Q < 0 \Rightarrow$ heat out of system

Example: pour $m_1 = 5g$ milk (basically water) @ T_1: 4° C into

$m_2 = 495g$ water @ T_2: 96° C



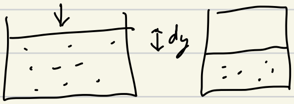
$$m_1 c_w (T_F - 4°C) + m_2 c_w (T_F - 96°C) = 0$$

$$\Rightarrow (m_1 + m_2) T_F = m_1 T_1 + m_2 T_2$$

$$T_F = \frac{m_1 T_1 + m_2 T_2}{m_1 + m_2} = 95.08°C$$

What kind of "Work" could we be talking about in the First law?

Consider a gas in a piston:



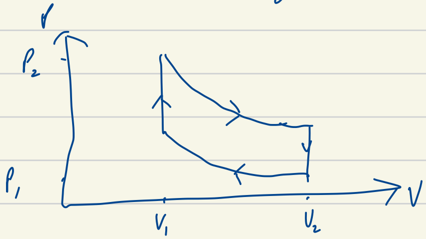
$$dW = -F d_y = -PA\, d_y = -PdV$$

work done on gas

Force through a displacement

$$\Rightarrow W = -\int_{V_{initial}}^{V_{Final}} P(V)\, dV$$

See text for work cycles:



Area enclosed by work cycles = work done by gas

Lecture 24: Thermodynamics, part 2

[Course evals - thanks!

We observe lots of processes that go "one way"

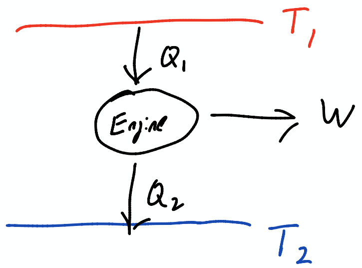
- Heat flows from hot to cold
- a sliding object experiences friction and will slow down and heat up
- a gas expands to fill a container

Notice: All of these going in reverse are consistent w/ first law and energy conservation, so.. why don't we see them?

It'll be a consequence of the second law.

Heat engine: A machine that takes heat from a source, transfers some of that energy to a reservoir and in the process does work (eg: describe a steam engine.

Schematic



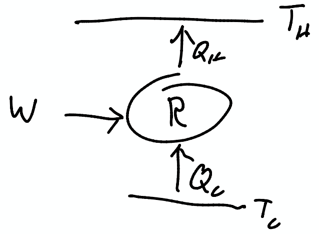
energy conservation : $Q_1 = Q_2 + W$

efficiency: $e = \frac{W_1}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$

$$0 \leq e \leq 1$$

Carnot engine : $e = 1 - \frac{T_2}{T_1}$, Carnot's theorem: No engines better than Carnot's

A "Refrigerator" is an engine running in reverse!



2nd Law: Kelvin: No perfect engines $(e < 1)$

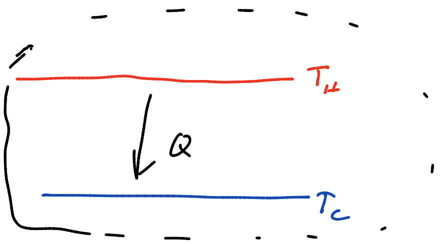
Clausius: No perfect refrigerators

Subtle Consequence! Just like 0th Law implied a new variable that characterizes an equilibrium system, the 2nd law implies that the entropy, S, is another variable describing thermodynamic systems.

For a reversible process, if a small amount of heat dQ is transferred to a system at temperature T, the change in entropy is

$$dS = \frac{dQ}{T}$$

example:



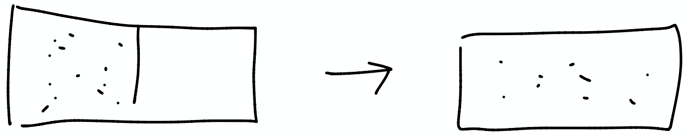
isolate and let heat flow:

$$\Delta S = \frac{Q}{T_C} + \frac{(-Q)}{T_H} = Q\left(\frac{1}{T_C} - \frac{1}{T_H}\right)$$

So: natural processes in which heat flows $\Rightarrow \Delta S > 0$

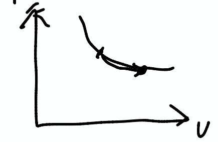
"Entropy" version of 2nd law: $S_{universe}$ always goes up.

Joule's Free expansion experiment:



P_1, V_1, T $\rightarrow$ $P_2, V_2, T_2 = T$

$T_1 = T_2$ because and $W=0$ (just remove a partition), so E_{int} is the same.



$$dS = \frac{dQ}{T} = \frac{PdV}{T} = NK_B \frac{T}{T}\frac{dV}{V} = Nk_B \frac{dV}{V}$$

integrate $\uparrow$ along isotherm:

$$\Delta S = NK_B \log\left(\frac{V_2}{V_1}\right)$$

let's interpret this as

$$\Delta S = K_B \log\left[\left(\frac{V_2}{V_1}\right)^N\right)$$

and think about V_m, the average volume of a molecule... the # of possible position states of a gas molecule is $\approx \frac{V}{V_m}$, so # of ways of putting N molecules in a container is

$$\Omega \approx \left(\frac{V}{V_m}\right)^N$$

then $\Delta S = K_B \log\left[\frac{V_2^N}{V_1^N}\right] = K_B \log\left[\frac{V_2^N}{V_m^N}\frac{V_m^N}{V_1^N}\right] = K_B \log\left(\frac{\Omega_2}{\Omega_1}\right)$

$= K_B \log \Omega_2 - K_B \log \Omega_1$

or, $S_{state} = K_B \log \Omega$

Boltzmann's Formula, connecting the microscopic world and the macroscopic world!

Boltzmann Grave; eddington quote, entropy demo.